Under CH, the boolean completion of a type III factor's projection lattice is the standard continuum-collapsing algebra

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When set-theoretic forcing is carried out with a measure algebra it is known as random-real forcing; since the projection lattice $\mathbb{P}$ of a type III von Neumann factor is a noncommutative analog of a measure algebra, forcing with $\mathbb{P}$ can be considered a noncommutative analog of random-real forcing. Does this forcing have any interesting, novel properties? We show here that, under the continuum hypothesis at least, it does not: the well-known poset conventionally used to force the continuum to become countable embeds densely into $\mathbb{P}$, so they are forcing-equivalent. (More precisely, their boolean completions are isomorphic.)

Definitions

$\mathcal{R}$ is a separably-acting type III von Neumann factor; $\mathbb{P}$ is its projection lattice,

$$\mathbb{P} = \{P \in \mathcal{R} : P^2 = P^* = P\},$$

with the usual ordering $P \leq Q \iff PQ = P$. The greatest and least members of $\mathbb{P}$ are the identity projection 1 and the null projection 0. $\mathbb{P}^+$ means $\mathbb{P} \setminus \{0\}$. When $A, B \in \mathbb{P}$, we write $A \perp B$ to mean they are orthogonal ($AB = 0$).

Partitions of $\mathbb{P}$ will be central to our arguments; by this we always mean a *lattice* partition, so a partition of $P \in \mathbb{P}^+$ is a maximal pairwise-disjoint subset of the projections in $\mathbb{P}^+$ that are $\leq P$. A partition is nontrivial if its cardinality is > 1.

A state ϕ on $\mathcal{R}$ is a complex linear functional on $\mathcal{R}$ that is normalized in the sense that $\phi(1) = 1$. A state is *faithful* if $\phi(T) = 0$ holds only when $T = 0$; it is *normal* if it is countably additive on sets of mutually orthogonal projections, in which case it is also continuous on $\mathbb{P}$ with respect to the strong operator topology.

For cardinals $\kappa > \aleph_0$, C_κ is the set of finite sequences of ordinals $< \kappa$, ordered by reverse inclusion; this is the standard poset used to force κ to become countable.

Recall that when A, B are posets, a *dense embedding* $\phi : A \to B$ is an order-isomorphism onto a subset of B such that every $b \in B$ has some $a \in A$ satisfying $\phi(a) \leq b$.

If one separative poset embeds densely into another, then the embedding extends naturally to an isomorphism between their boolean completions. Thus to show $B(C_\kappa)$ isomorphic to $B(\mathbb{P})$, it suffices to show that C_κ embeds densely into $\mathbb{P}^+$.

1 The main argument, excluding the proof that $\mathbb{P}$ has no countable nontrivial partition

Lemma 1.1 *Suppose that for some faithful normal state ϕ on $\mathcal{R}$, and all $P \in \mathbb{P}^+$, and all $\epsilon > 0$, there exists a lattice partition X of P such that (i) $|X| = 2^{\aleph_0}$, (ii) $Q \in X \Rightarrow \phi(Q) < \epsilon$, and (iii) for all $P' \in \mathbb{P}^+$, $P' \leq P$, there exists $Q \in X$ such that either $P' \leq Q$ or $Q \leq P'$. Then there is a dense embedding of $C_{2^{\aleph_0}}$ into $\mathbb{P}^+$.*

Let $P_{\langle\rangle}$ (indexed by the empty sequence) be $1 \in \mathbb{P}$. Let $\{P_{\langle\alpha\rangle} : \alpha < 2^{\omega_0}\}$ be a lattice partition of $P_{\langle\rangle}$ of the kind supposed, with $\epsilon = 1/2$. Next, for each α obtain a lattice partition $\{P_{\langle\alpha,\beta\rangle} : \beta < 2^{\omega_0}\}$ of $P_{\langle\alpha\rangle}$, with $\epsilon = 1/4$. Continue in this manner so that for every finite sequence $\vec{\alpha}$ of ordinals $< 2^{\omega_0}$, $\phi(P_{\vec{\alpha}}) < 2^{-|\vec{\alpha}|}$. Clearly $\vec{\alpha} \mapsto P_{\vec{\alpha}}$ is an embedding of $C_{2^{\aleph_0}}$ into $\mathbb{P}^+$; we must confirm that it is a dense embedding.

For $n > 0$, define $X_n := \{P_{\vec{\alpha}} : |\vec{\alpha}| = n\}$. Each X_n is then a lattice partition of $1 \in \mathbb{P}$ satisfying (i), (ii), (iii) for $\epsilon = 2^{-n}$. Now given $P \in \mathbb{P}^+$, fix n such that $2^{-n} < \phi(P)$; then by (iii), some $P_{\vec{\alpha}} \in X_n$ satisfies either $P \leq P_{\vec{\alpha}}$ or $P_{\vec{\alpha}} \leq P$; but the first alternative is excluded because each $P_{\vec{\alpha}} \in X_n$ satisfies $\phi(P_{\vec{\alpha}}) < 2^{-n} < \phi(P)$. □

Lemma 1.2 *If, for all $P \in \mathbb{P}^+$, every nontrivial lattice partition of P has cardinality $2^{\aleph_0}$, then there is a dense embedding of $C_{2^{\aleph_0}}$ into $\mathbb{P}^+$.*

Fix $P \in \mathbb{P}^+$, $\epsilon > 0$, and a faithful normal state ϕ on $\mathcal{R}$; it suffices to show there exists a lattice partition X of P meeting requirements (i), (ii), (iii) of Lemma 1.1.

Enumerate as $\{P_\alpha : \alpha < 2^{\aleph_0}\}$ the projections in $\mathbb{P}^+$ that are $< P$.

We define Q_α by induction on α, for all $\alpha < 2^{\aleph_0}$, with the induction hypothesis that the previously-defined Q_β are pairwise-disjoint. At step α, if $P_\alpha \leq Q_\beta$ for some $\beta < \alpha$, let $Q_\alpha = 0$. Otherwise, under the present lemma's supposition, $\{Q_\beta \wedge P_\alpha : \beta < \alpha\}$ cannot be a partition of P_α (in the lattice $\mathbb{P}$), because its cardinality is less than $2^{\aleph_0}$. Thus $\{Q_\beta \wedge P_\alpha : \beta < \alpha\}$ is a pairwise-disjoint but not *maximal* pairwise-disjoint subset of the projections in $\mathbb{P}^+$ below P_α; so there exists in $\mathbb{P}^+$ a projection $< P_\alpha$ that is disjoint from all Q_β, $\beta < \alpha$. Let Q_α be such a projection satisfying $\phi(Q_\alpha) < \epsilon$ (it follows from the fact that these projections form a downward-closed subset of $\mathbb{P}^+$, which has no minimal projections, and from the normality of ϕ, that we can require an arbitrarily small ϕ value here).

Let X be the set of all the nonzero Q_α so obtained. Clearly requirements (ii) and (iii) of Lemma 1.1 are met, and X is pairwise-disjoint. Moreover X must be a *maximal* pairwise-disjoint subset of the projections in $\mathbb{P}^+$ below P — i.e. a partition of P — because any such projection we might hope to add to X was already enumerated as one of the P_α, and either P_α was $\leq Q_\beta$ for some $\beta < \alpha$ or $P_\alpha \wedge Q_\alpha \neq 0$. Finally, by supposition, X has the cardinality required by (i). □

Corollary 1.3 *If CH holds and $\mathbb{P}$ has no countable nontrivial partition then there is a dense embedding of $C_{2^{\aleph_0}}$ into $\mathbb{P}^+$.*

CH means that $2^{\aleph_0}$ is $\aleph_1$, the smallest uncountable ordinal. This plus the supposition that $\mathbb{P}$'s nontrivial partitions are uncountable implies they all have cardinality $2^{\aleph_0}$ (note $|\mathbb{P}| \leq |\mathcal{R}| \leq |\mathcal{B}(H)|$, which is $\leq 2^{\aleph_0}$ when H is separable, because $\mathcal{B}(H)$ then has a countable dense subset in the strong operator topology, from which at most $2^{\aleph_0}$ convergent nets can be chosen). And this, plus the observation that $\{P' \in \mathbb{P} : P' \leq P\}$ is order-isomorphic to all of $\mathbb{P}$ for any $P \in \mathbb{P}^+$, implies the supposition of Lemma 1.2. □

2 Each nontrivial lattice partition of a type III $\mathbb{P}$ is uncountable

Our strategy is to take any countable subset $\{P_n\} \subseteq \mathbb{P}^+ \setminus \{1\}$ and show it is not a lattice partition, by finding $Q \in \mathbb{P}^+$ disjoint from all the P_n.

In order to argue geometrically we assume $\mathcal{R}$ to act on a (separable) Hilbert space H.

We will construct Q as the limit of approximating projections $W_n W_n^*$, where the W_n are partial isometries. Recall that a partial isometry W maps its initial space (the range of the projection W^*W) isometrically onto its final space (the range of the projection WW^*), and annihilates the orthogonal complement of its initial space.

When V is a partial isometry and $Y \leq V^*V$ is a projection, we call VY a *restriction* of V; VY is a partial isometry whose action agrees, on its initial space, with V's .

We fix a faithful normal state ϕ on $\mathcal{R}$, to keep track of the "size" of the sequence of projections $W_n^* W_n$, so that they do not vanish in the limit.

For all $m \geq 0$ we wish to keep $W_n W_n^*$ sufficiently separated from P_m, for all $n > m$, so that in the limit we will have $Q \wedge P_m = 0$. We formalize this by using $\|P_m W_n W_n^*\|$ to quantify the "proximity" of $W_n W_n^*$ to P_m: if $\|P_m W_n W_n^*\| < 1$ then $(W_n W_n^*) \wedge P_m = 0$ (but the converse need not hold).

We begin with two geometric lemmas.

Lemma 2.1 *Suppose* $A, K, Z \in \mathbb{P}^+$ *and* $P_0, D \in \mathbb{P}$ *satisfy* $A \perp Z$, $P_0 \perp (A+Z)$, $D \perp (A+Z)$, $K \leq (A+Z)$; *then* $\|P(K+D)\| \leq \max(\|PK\|, \|PD\|)$, *where* P *denotes* $(A+P_0)$.

We affirm first that $(A+Z)$, $P = (A+P_0)$, and $(K+D)$ are all projections, thanks to the given orthogonality relations.

Let v be an arbitrary unit vector in $\mathrm{ran}(K+D)$; it suffices to show that there exists either unit $k \in \mathrm{ran}(K)$ such that $\|Pk\| \geq \|Pv\|$, or unit $d \in \mathrm{ran}(D)$ such that $\|Pd\| \geq \|Pv\|$.

It follows from $v \in \mathrm{ran}(K+D)$, $\|v\| = 1$, and $K \perp D$, that there exist unit $k \in \mathrm{ran}(K)$, unit $d \in \mathrm{ran}(D)$, and $r \in [0,1]$ such that $v = rk + \sqrt{1-r^2}d$. Therefore

$$Pv \;=\; P(rk + \sqrt{1-r^2}d) \;=\; rPk + \sqrt{1-r^2}Pd. \qquad (\star)$$

We now claim $Pk \in \mathrm{ran}(A)$ and $Pd \in \mathrm{ran}(P_0)$. In the first case we use $K \perp P_0$ (which is clear from the given relations $K \leq (A+Z)$ and $P_0 \perp (A+Z)$). Since $Pk = Ak + P_0 k$, and

$k \in \operatorname{ran}(K)$, we have $Pk = Ak + 0 = Ak \in \operatorname{ran}(A)$. $Pd \in \operatorname{ran}(P_0)$ is shown similarly. Now since $A \perp P_0$, we may apply the Pythagorean theorem to $(\star)$:

$$\|Pv\|^2 = r^2\|Pk\|^2 + (1-r^2)\|Pd\|^2. \tag{$\star\star$}$$

To find the $r \in [0,1]$ giving the maximum value to the expression on the right side of $(\star\star)$, we set its derivative to zero to find critical points:

$$\begin{aligned}\frac{d}{dr}(r^2\|Pk\|^2 + (1-r^2)\|Pd\|^2) &= 2r\|Pk\|^2 - 2r\|Pd\|^2 \\ &= 2r(\|Pk\|^2 - \|Pd\|^2) = 0.\end{aligned}$$

We see that if $\|Pk\| \neq \|Pd\|$ then there are no solutions (hence no critical points) for r-values strictly between 0 and 1, so the maximum value must be attained at $r = 0$ or $r = 1$. The maximum will be attained there too if $\|Pk\| = \|Pd\|$ since in that case the expression in $(\star\star)$ takes the same value for all $r \in [0,1]$. In the case that $r = 0$ yields the maximum value, we deduce $\|Pd\|^2 \geq \|Pv\|^2$, and in the $r = 1$ case, $\|Pk\|^2 \geq \|Pv\|^2$. Thus k or d will serve as the unit vector it sufficed to find. □

Lemma 2.2 *If $P \in \mathbb{P}^+$ and $V_0, V \in \mathcal{R}$ are non-null partial isometries with the same initial space (i.e. $V^*V = V_0^*V_0$) then $\|PVV^*\| \leq \|PV_0V_0^*\| + \|V - V_0\|$.*

Since V and V_0 are partial isometries with the same initial space, the unit vectors in $\operatorname{ran}(VV^*)$ are those vectors having form Vw for some unit vector $w \in \operatorname{ran}(V_0^*V_0)$. For all such w, both V_0w and Vw are unit vectors, and we have $Vw = V_0w + x$ for some x satisfying $\|x\| \leq \|V - V_0\|$. Thus:

$$\begin{aligned}\|PVV^*\| &= \sup\{\|PVw\| : w \in \operatorname{ran}(V_0^*V_0), \|w\| = 1\} \\ &\leq \sup\{\|P(V_0w + x)\| : w \in \operatorname{ran}(V_0^*V_0), \|w\| = 1, \|x\| \leq \|V - V_0\|\} \\ &\leq \sup\{\|PV_0w\| + \|Px\| : w \in \operatorname{ran}(V_0^*V_0), \|w\| = 1, \|x\| \leq \|V - V_0\|\} \\ &\leq \sup\{\|PV_0w\| : w \in \operatorname{ran}(V_0^*V_0), \|w\| = 1\} + \|V - V_0\| \\ &= \|PV_0V_0^*\| + \|V - V_0\|.\end{aligned}$$

(In the third line we have used a triangle inequality, and in the fourth line the fact that P, being a projection, cannot increase the norm of x, which is at most $\|V - V_0\|$.) □

Lemma 2.3 (Decomposition lemma) *Given $\epsilon > 0$ and $P \in \mathbb{P}^+$, any partial isometry $W \in \mathcal{R}$ can be written as a sum $W = A + B + C$ of partial isometries in $\mathcal{R}$ whose initial spaces are mutually orthogonal, such that:*
(1) $AA^* = WW^* \wedge P$;
(2) $\phi(B^*B) < \epsilon/2$;
(3) $\|PCC^*\| < 1$.

We will show this using the spectral theory for self-adjoint operators in a von Neumann algebra; see Theorems 5.2.2 and 5.2.3 of [2], whose notation we follow.

Consider the operator $PW \in \mathcal{R}$. The basic idea is to take the polar decomposition of PW, and then consider the spectral resolution of the self-adjoint part.

The polar decomposition of PW is a pair U, T of operators such that $PW = UT$, with T a self-adjoint operator having spectrum in $[0, \infty)$, and U a partial isometry whose initial space is the closure of T's range. By [2], Proposition 6.1.3, we may assume U, T are a polar decomposition of PW such that $U, T \in \mathcal{R}$.

Since U is norm-preserving on its initial space, which is the closure of $\operatorname{ran}(T)$, we have $\|UTv\| = \|Tv\|$ for all v. From this and $UT = PW$ we conclude

$$\|Tv\| = \|PWv\|, \quad v \in H. \tag{*}$$

Suppose first that $\|T\| < 1$. Since (*) entails $\|T\|$ is an upper bound to $\|PWv\|$ for unit v, it is an upper bound for $\|PWW^*\|$. In this case we simply set $A = B = 0$, and $C = W$.

Now consider the remaining case $\|T\| = 1$. Let $r(T)$ denote T's right support projection (whose range is the orthogonal complement of T's kernel); note $r(T) \leq W^*W$. Consider the spectral resolution of T, i.e. the projections $E_\lambda \in \mathbb{P}$ (for all real λ) such that:

$$\begin{aligned} &\text{For } \gamma \leq \lambda, E_\gamma \leq E_\lambda; \\ &\text{For } \gamma < \lambda \text{ and all unit } v \in \operatorname{ran}(E_\lambda - E_\gamma), \|Tv\| \in (\gamma, \lambda]; \\ &\text{Because } T \text{ has spectrum in } [0, \infty) \text{ and } \|T\| = 1, E_1 - E_0 = r(T). \end{aligned} \tag{**}$$

Let $E_{<1}$ denote $\bigvee_{\gamma<1} E_\gamma$; note we may or may not have $E_{<1} = E_1$.

For all $\lambda \in (0, 1)$, the following is then an orthogonal decomposition of $r(T)$:

$$r(T) = (E_1 - E_{<1}) + (E_{<1} - E_\lambda) + (E_\lambda - E_0).$$

Furthermore, since $r(T) \leq W^*W$, the following is also an orthogonal decomposition:

$$W^*W = (E_1 - E_{<1}) + (E_{<1} - E_\lambda) + (W^*W - (E_1 - E_\lambda)).$$

As $\lambda \to 1$, the projections $(E_{<1} - E_\lambda)$ converge to the null projection in the strong operator topology. Since ϕ is a normal state it is continuous on $\mathbb{P}$ in the s.o.t., so $\phi(E_{<1} - E_\lambda)$ will converge to 0. So we may fix λ sufficiently close to 1 that $\phi(E_{<1} - E_\lambda) < \epsilon/2$. (It is fine if $E_{<1} - E_\lambda = 0$; in this case we will just have $B = 0$.)

Now define A, B, C to be the following restrictions of W:

$$A := W(E_1 - E_{<1}); \quad B := W(E_{<1} - E_\lambda); \quad C := W(W^*W - (E_1 - E_\lambda)).$$

Note this entails $A^*A = (E_1 - E_{<1})$; $B^*B = (E_{<1} - E_\lambda)$; $C^*C = (W^*W - (E_1 - E_\lambda))$.

The lemma's requirement (1) follows from the fact that unit v satisfies $Wv \in \operatorname{ran}(P)$ if and only if $\|Tv\| = 1$ (using (*), (**), and $A^*A = (E_1 - E_{<1})$).

Our choice of λ ensured requirement (2), $\phi(B^*B) < \epsilon/2$.

Requirement (3) follows from (*), (**), and $C^*C = (W^*W - (E_1 - E_\lambda))$. $\square$

The following lemma will be the core of the inductive step to obtain W_{n+1} from W_n.

Lemma 2.4 *Given $\epsilon \in (0,1)$, $P \in \mathbb{P}^+ \setminus \{1\}$, and a non-null partial isometry $W \in \mathcal{R}$, there exists a partial isometry $V \in \mathcal{R}$ such that*
(i) $V^*V \leq W^*W$;
(ii) $\phi(W^*W) - \phi(V^*V) < \epsilon$;
(iii) $\|WV^*V - V\| < \epsilon/3$
(iv) $\|PVV^*\| < 1$.

Invoke Lemma 2.3 to obtain partial isometries $A, B, C \in \mathcal{R}$, with mutually orthogonal initial spaces, whose sum is W; define $V_0 = A + C$.

V_0 now satisfies (i), (ii), and (iii). In particular $V_0 = WV_0^*V_0$; and because $\phi(B^*B) < \epsilon/2$ and states are additive on orthogonal projections, we have $\phi(W^*W) - \phi(V_0^*V_0) < \epsilon/2$.

If V_0 satisfies (iv), set $V = V_0$ and we are done; so assume $\|PV_0V_0^*\| = 1$. By Lemma 2.1 and clause (iii) of Lemma 2.3, and $V_0^*V_0 = A^*A + C^*C$, we have $A \neq 0$. Note $AA^* \leq P$.

We now set aside a projection Z, orthogonal to both P and CC^*, towards which we will rotate AA^*.

Let $Z \in \mathbb{P}^+$ be an arbitrary projection $\leq (1 - P)$ such that, when Y denotes the projection onto the closed range of CC^*Z, we have $\phi(C^*YC) < \epsilon/2$. (It is fine if $Y = 0$. As noted in Lemma 1.2, it is legitimate to demand an arbitrarily small ϕ-value.) Let D be the restriction of C, $D := C(1 - C^*YC)$ (equivalently, $D := (1 - Y)C$). Note we have $DD^* \perp Z$ and $\|PDD^*\| \leq \|PCC^*\| < 1$, and $D^*D = C^*C - C^*YC$.

Let $V_1 = A + D$ (this is a restriction of V_0, hence too of W). This V_1 satisfies (i), and since $\phi(C^*C) - \phi(D^*D) < \epsilon/2$, we have $\phi(W^*W) - \phi(V_1^*V_1) < \epsilon$, so it satisfies (ii) as well.

We now invoke the Murray-von-Neumann equivalence of all non-null projections in a type III factor to obtain a partial isometry $K \in \mathcal{R}$ whose initial space is A's final space (i.e. $K^*K = AA^*$) and whose final space is Z's range (i.e. $KK^* = Z$). Since $AA^* \leq P$ and $Z \perp P$, we have $K^*K \perp KK^*$. Therefore K_ϵ defined as

$$K_\epsilon := \cos(\epsilon/3)AA^* + \sin(\epsilon/3)K,$$

is a partial isometry, and can be described as a partial rotation by $\epsilon/3$ radians of A's final space towards Z's range. $K_\epsilon A$ is also a partial isometry; its final-space projection $K_\epsilon AA^*K_\epsilon^*$ is equal to $K_\epsilon K_\epsilon^*$, which can be described as a subprojection of P that has been rotated towards P's orthogonal complement. It follows that $\|PK_\epsilon AA^*K_\epsilon^*\| < 1$.

Set $V = K_\epsilon A + D$. Note first that $V^*V = V_1^*V_1$, so (i) and (ii) are still satisfied. For requirement (iv) we appeal to Lemma 2.1, using our AA^* for that lemma's projection A, our $K_\epsilon K_\epsilon^*$ for its projection K, our DD^* for its projection D, our Z for its Z, and our $(P - AA^*)$ for its P_0. We conclude

$$\|PVV^*\| = \|P(K_\epsilon AA^*K_\epsilon^* + DD^*)\| \leq \max(\|P(K_\epsilon AA^*K_\epsilon^*)\|, \|PDD^*\|) < 1,$$

so requirement (iv) is satisfied.

It remains to verify (iii). Consider the orthogonal decomposition of H into three subspaces, the ranges of D^*D, of A^*A, and of $(1 - V^*V)$. It suffices to show that for unit v in each subspace, $\|WV^*Vv - Vv\| \leq \epsilon/3$. All vectors in $\mathrm{ran}(1 - V^*V)$ are annihilated by both V and WV^*V. For the other two subspaces we have $V^*Vv = v$, so it suffices to show $\|Wv - Vv\| < \epsilon/3$. On vectors in $\mathrm{ran}(D^*D)$, V's action is the same as W's. For unit $v \in \mathrm{ran}(A^*A)$, $\|Wv\| = 1$, and Vv is Wv rotated by $\epsilon/3$ radians; therefore by simple trigonometry, $\|Wv - Vv\| < \epsilon/3$. □

Lemma 2.5 *When $\mathbb{P}$ is the projection lattice of a type III von Neumann factor $\mathcal{R}$, each nontrivial lattice partition of $\mathbb{P}$ is uncountable.*

Let $\{P_n : n < \omega\} \subseteq \mathbb{P}^+ \setminus \{1\}$ be arbitrary. We will find $Q \in \mathbb{P}^+$ that is disjoint from all the P_n. Thus even if $\{P_n\}$ is a pairwise-disjoint subset of $\mathbb{P}^+$, it is not a *maximal* such subset, i.e. not a partition of $\mathbb{P}$. In order to avoid discussing separately the case of finite sets $\{P_n : n < m\}$, we allow the same projection to be indexed by many (possibly infinitely many) n in $\{P_n\}$.

We will define Q using a sequence $\{W_n\}$ of partial isometries in $\mathcal{R}$, and a decreasing sequence $\{\epsilon_n\}$ of reals.

Set $W_0 := P_0$, and $\epsilon_0 := \phi(P_0)/2$. We want the following to hold for $n \geq 0$:

— (0) $0 < \epsilon_{n+1} < \epsilon_n/3$;

— (1) $W_{n+1}^* W_{n+1} \leq W_n^* W_n$;

— (2) $\phi(W_n^* W_n) - \phi(W_{n+1}^* W_{n+1}) < \epsilon_n$;

— (3) $\|W_{n+1} - W_n W_{n+1}^* W_{n+1}\| < \epsilon_n/3$;

— (4) $\|P_n W_{n+1} W_{n+1}^*\| \leq 1 - \epsilon_{n+1}$.

To fulfill these conditions we iterate over $n \geq 0$. At stage n, let W_{n+1} be the partial isometry obtained from Lemma 2.4 using W_n for its V and ϵ_n for its ϵ. Then conditions (1), (2), (3) will hold by the corresponding clauses of Lemma 2.4. Set ϵ_{n+1} to the lesser of $\epsilon_n/4$ and $1 - \|P_n W_{n+1} W_{n+1}^*\|$; then conditions (0) and (4) hold.

We now show how to use the W_n and ϵ_n to define Q as required.

From (1), the sequence $\{W_n^* W_n\}$ of projections onto initial spaces is decreasing, and so it converges to a projection X with respect to $\mathcal{B}(H)$'s strong operator topology (see e.g. Remark 2.5.9 of [2]). Note $X \in \mathbb{P}$ since $\mathcal{R}$, being a von Neumann subalgebra of $\mathcal{B}(H)$, is closed in this topology. To show $X > 0$, note first that

$$\sum_{n \geq 0} \epsilon_n \;\leq\; \frac{\phi(P_0)}{2} \sum_{n \geq 0} 3^{-n} \;=\; \frac{3\phi(P_0)}{4}.$$

Then by (2), and the fact (see Theorem 7.1.12 of [2]) that normal states (like ϕ) are strong-operator-continuous on $\mathbb{P}$, we have

$$\phi(X) \geq \phi(P_0) - 3\phi(P_0)/4 = \phi(P_0)/4,$$

so X is not null.

We will now use the fact (mentioned at the outset of Section 2) that a restriction VY of a partial isometry V is a partial isometry that agrees with V on its initial space. This fact along with (1) implies that the operators W_{n+1} and $W_n W_{n+1}^* W_{n+1}$ referenced in (3) are both partial isometries; the latter has the same initial space as W_{n+1}. The projection

X just defined is the projection onto a subspace of this initial space. It follows that for all n, the restriction $W_n X$ is a partial isometry whose initial space is $\mathrm{ran}(X)$.

It is then straightforward to verify from (3) that

$$\|W_{n+1}X - W_n X\| < \epsilon_n/3; \tag{$\dagger$}$$

$$\|W_{n+1}XW_{n+1}^* - W_n X W_n^*\| < \epsilon_n/3. \tag{$\ddagger$}$$

From (0) and (†) we conclude that the partial isometries $W_n X$ converge in norm to a partial isometry $W \in \mathcal{R}$. Since $W^*W = X$, $W \neq 0$. The final-space projections $W_n X W_n^*$ also converge in norm, to $WW^* \in \mathbb{P}^+$. Let Q be WW^*. It remains to show that $Q \wedge P_n = 0$ for all n.

Fix n. Since the projection $W_{n+1}XW_{n+1}^*$ is $\leq W_{n+1}W_{n+1}^*$, the former's proximity to P_n cannot be greater than the latter's proximity to P_n; thus by (4),

$$\|P_n W_{n+1} X W_{n+1}^*\| \leq 1 - \epsilon_{n+1}.$$

For all $m > n$, (‡) and Lemma 2.2 imply

$$\|P_n W_{m+1} X W_{m+1}^*\| < \|P_n W_m X W_m^*\| + \epsilon_m/3.$$

Since Q is the norm limit of the $W_m X W_m^*$, the two last inequalities imply

$$\|P_n Q\| \leq 1 - \epsilon_{n+1} + \sum_{m=n+1}^{\infty} \epsilon_m/3.$$

By (0), the sum on the right is $\leq \epsilon_{n+1}(1/3 + 1/9 + \ldots) = \epsilon_{n+1}/2$, so we have $\|P_n Q\| \leq 1 - \epsilon_{n+1}/2$, implying $Q \wedge P_n = 0$.

Thus Q witnesses the failure of $\{P_n\}$ to be a maximal pairwise-disjoint subset of $\mathbb{P}^+$, i.e. its failure to be a partition of $\mathbb{P}$. □

Theorem 2.6 *CH implies $B(\mathbb{P}) = B(C_{\aleph_1})$, i.e. $\mathbb{P}$'s boolean completion is isomorphic to that of $C_{\aleph_1}$.*

It follows from Lemma 2.5, Corollary 1.3, and CH that there is a dense embedding of $C_{\aleph_1}$ into $\mathbb{P}^+$. As we noted above, this embedding extends naturally to a boolean isomorphism of $B(\mathbb{P})$ onto $B(C_{\aleph_1})$. □

Question: Does "All nontrivial partitions of $\mathbb{P}$ have cardinality $2^{\aleph_0}$" follow from ZFC alone (without CH)? If so, it would follow from Lemma 1.2 that $\mathbb{P}$ is always forcing-equivalent to the standard continuum-collapse poset, independently of the choice of ambient ZFC model.

References

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