Overview of the Bergsonian axioms project

Doug McLellan, June 2024

The Bergsonian axioms project is an effort to formalize the eponymous philosopher's ideas into a scientifically respectable alternative to belief in "possible worlds," and so to evade the depressing consequences of that belief. The axioms define a self-constructing mathematical universe in which new possibilities grow in a continuous, causal, yet non-deterministic way. This would clarify and lend support to Bergson's impressionistic account of freedom through "creative evolution" — if a structure satisfying the axioms could be found. This has not yet happened; this overview is partly a motivational speech for continuing the search.

The possible-worlds framework

By "the possible-worlds framework" we mean this assertion: "Mathematics, which is independent of time, delineates all the possible trajectories for everything that exists, and our world is one of these possible trajectories." Note that this is compatible with different ways of thinking about our own world's apparently special status: there could exist some good explanation why our world alone is real, or it could be a matter of luck, or else maybe all the possibilities are equally real. This framework (in its "good explanation" version) had its first great champion in Leibniz. Over the centuries its prominence has occasionally waned, but it has waxed recently to the point that it is now, arguably, the default metaphysical view of physicists and philosophers in the analytic tradition.

The possible-worlds framework thrives today not only in academia but in the popular culture, where its "all-possible-worlds-are-equally-real" version is especially trendy. On the street, one hears things like "There's no possible world where I agree to date you!" At the cinema, one is treated to the dramatic possibilities of the "Multiverse." On the other hand, the recent movie *Everything Everywhere All At Once* shows how the Multiverse idea threatens to put a stop to all drama, in a sense. If fictional characters — or if we ourselves — can be said to make every possible choice at every branching point in life, nobody can truly be said to choose anything. The demoralizing conclusion seems to be that true human freedom and agency are impossible. (Anyone who has seen the movie will recall how it highlights this conclusion by showing all the possible worlds compressed into a giant soul-crushing bagel.)

We concur with this conclusion, but we would like to stress that it would follow not just from the reality of all possible worlds: it follows from the possible-worlds framework in general. Switching to a version of the framework in which only one possible world is real would not, we argue, rescue our freedom and agency. Consider: in the version where some good explanation determines which world is real, it would be that explanation that ultimately determines our actions; and in the version where random luck determines which world is real, chance would determine our actions. In no case can we enjoy anything worthy of the term "free will."

(To be clear, we have no wish to *convince* anyone that the possible-worlds framework is depressing, and there are of course writers who celebrate it: The late D. Dennett, to take one example, argued entertainingly that by "freedom" we must just have meant a very nifty form of determinism all along, and we wish Godspeed to anyone who finds such arguments satisfying.)

Bergson's fuzzy alternative: Life as possibility-creation

It has proven difficult, alas, to come up with alternatives to the possible-worlds framework that are compatible with modern science. Henri Bergson's attempt, in the early 20th century, stands out as the most subtle, substantial, and creative. His key insight was that life is not the successive actualization of pre-existing possibilities, but rather the continuous creation of novel possibilities. There is, Bergson claimed, no fixed totality of ways the world could be; instead, we help build a world that is *radically new* at each moment. *"Of a wholly new action ... which does not preexist its realization in any way, not even in the form of pure possibility, [philosophers] seem to have had no inkling. Yet that is just what free action is."*[1]

Bergson briefly towered over the intellectual scene of his time to a degree that seems, from our viewpoint across a century, scarcely believable. From this same viewpoint, which of course sits atop the Everest of scientific advances that the century has accumulated, Bergson's fall from influence seems attributable mainly to his failure to integrate his ideas rigorously with science. Bertrand Russell, the self-appointed rigor sheriff of Bergson's day, mocked his work as impressionistic and overly metaphorical. Russell was at least correct that physics and mathematics could get little purchase on Bergson's "new possibilities," which were generally portrayed as psychological, artistic, or at most biological. And Bergson did not help his cause by quarreling publicly with Einstein, putting into relief the gap between his own ideas and hard science. We do not mean to suggest here that Bergson's work is worthless unless that gap is bridged — only that, without some kind of scientific formalization, it will remain where it has rested since the 1930s, off to the side of mainstream intellectual life.

"Mathematizing" Bergson's alternative

The Bergsonian axioms project is an attempt to formalize the insight just described. It seeks to understand Bergson's "newly arising possibilities" as new *mathematical* possibilities.

[1] From Bergson's introduction to [1]. This collection includes his late essay "Le possible et le réel," which he intended to present at his acceptance of the Nobel prize for literature, until he was prevented by poor health; this essay beautifully distills the core ideas that our project seeks to formalize.

Recall the possible-worlds framework's claim that "mathematics, which is independent of time, delineates all the possible trajectories for everything that exists." If mathematics were not independent of time — if, instead, the mathematical universe *grew* over time — the foundation of the possible-worlds framework would be kicked out from under it. As a first approximation of what we have in mind here, imagine a quantum measurement that yielded, rather than a rational value like 1.03 centimeters, a non-repeating, non-terminating decimal value that began 1.039466746 . . . and was moreover *not even abstractly or virtually possible before being instantiated by this measurement.*

While it may sound strange to speak of new real numbers becoming possible, and while it would indeed have made little sense in Bergson's day, the idea is now commonplace in the mathematical domain of set theory. The set-theoretic method of *forcing*, which dates from the 1960s, starts by positing the existence of the totality of mathematically possible structures, and then proves it is logically consistent that new "generic" structures with a wide variety of properties could be added to this totality. In most cases the new structures are new infinite sequences of digits, or, equivalently, new real numbers. And to be clear, these generic sequences cannot be definable, as the decimal expansions of $3/17$ or 2π are; they must be, in the precise sense established by Gödel, *unconstructible.*

The emergence of new real numbers through forcing is generally imagined as something that happens just once, as though by a flick of a mathematical wand, or else as something that happens in discrete steps ("iterated forcing"). So imagined, it would seem poorly suited to the philosophical concerns that motivate us here, for if we worry that life is nothing but a succession of cosmic dice rolls, it is small consolation to hear that the numbers on the dice happen to fall into a pattern that was not abstractly possible beforehand.

What we need is a way to see new mathematical possibilities growing in a continuous, gapless, organic way. Rather than just "appearing," the new real numbers at each moment should be created, by simple definable operations, from the totality of real numbers that had become possible up to that moment. These thoughts lead to the concept of a *Bergson history.*[2]

The axioms defining a Bergson history are the foundation of our project. They are precise rules, stated in the terminology of set theory, that a collection of "points" must satisfy in order to embody a mathematical kind of creative evolution. Each of the points that together constitute a Bergson history can be considered a snapshot of the mathematical universe's growth, or more formally as a continuum (meaning a totality of real numbers[3]). A Bergson history is self-constructing in the sense that nothing outside of it guides or feeds its growth: nothing is given in advance, each point is just the closure under definable operations of the collection of all real numbers that emerged at previous points, and a new point will emerge when and only when "vanishingly little work" would be required for those real numbers to "self-collect" into this new point.

[2]For mathematical details see [7], which uses the term "self-constructing continuum" for a structure that satisfies our core axioms, and reserves the term "Bergson history" for a structure that also collapses cardinals as described in the discussion of Russell's paradox below.

[3]More precisely, we mean by a "continuum" the set of all real numbers in some model of ZF, Zermelo-Frankel set theory.

The search for a model of the axioms

The question, then, is whether a Bergson history — defined as a mathematical structure that satisfies our axioms — can exist. Set-theoretic forcing is the technique suited to producing a Bergson history, but as we stated, we need a variety of forcing that produces increasingly "generic" real numbers in a gradual/continuous way, rather than in discrete steps.

Forcing uses a template called a Boolean algebra to determine the properties that the new generic sets will have. Some Boolean algebras can indeed produce real numbers in a gradual/continuous way. This necessary condition is, however, not sufficient to yield a Bergson history, and we have shown that the best-known examples of these algebras (Cohen algebras and measure algebras) cannot do so. This is because they suffer from a sort of extreme profusion of symmetries that we have called "flexible homogeneity."[4] At the other end of the spectrum from the flexibly homogeneous algebras are the so-called "rigid" Boolean algebras, which have no symmetries. We have used up huge quantities of time and coffee trying to build a rigid Boolean algebra that yields new real numbers in a gradual/continuous way, but without success. We now consider it more likely that a Bergson history will be produced from a Boolean algebra that (1) is not rigid, (2) is not flexibly homogeneous, and (3) produces increasingly "generic" real numbers in a gradual/continuous way. The search for such an algebra is the main goal of the Bergsonian axioms project today.

Applying the Bergsonian axioms to physics

What we have just described, the core of our project, is a search for a mathematical structure — one that would provide a picture of how our real universe might work, as an alternative to the possible-worlds framework. But it is difficult, admittedly, to imagine mapping a mathematical structure that satisfied the Bergsonian axioms onto our current theories of physics. The obvious approach would be to identify each space-time point with a distinct continuum of real numbers, in such a way that more real numbers are possible at later points than at earlier ones. To achieve this with forcing, we would associate each space-time point with a subalgebra of some outermost Boolean algebra (considered as a template for forcing), in such a way that, given an earlier point and a later point, the earlier point's subalgebra is always nested within the later point's.

Beyond the mathematical difficulty of finding a suitable forcing algebra for this, the scientific difficulty is that our whole setup seems alien to any actual theory of physics. Or, at least, to any physics that an undergraduate is likely to encounter. Algebraic quantum field theory (AQFT) does have similarities to our setup. It too associates regions of space-time with nested algebras, taken to represent the observables within those regions. These are "type III von Neumann algebras" rather than Boolean algebras, but algebras of the latter kind can be derived from them in a natural way, as Boolean completions of their projection lattices.

[4] This is shown in [7].

Investigating this resemblance between AQFT and our Bergson history setup, we were able, in [6], to characterize what happens when AQFT's projection lattices are used for set-theoretic forcing. Namely, the generic sets produced can quite naturally be seen as random states on the quantum systems that the lattices encode. This is intriguing in light of the apparent randomness of states that result from quantum measurement; if this idea could be fleshed out, it could help us understand why quantum randomness must be part of our world.

An even more intriguing possible link between AQFT and Bergson histories was R. Longo's discovery [2] that type III von Neumann algebras can be "rigidly included" within each other. This discovery, combined with some results of M. Mori in [9], entails that the corresponding projection lattices will also be rigidly included within each other. It seemed plausible in light of this that these lattices' Boolean completions would then be, if not rigidly included, then at least not flexibly homogeneous. They might thus be uniquely suited to generate Bergson histories. Here, alas, is where our project has hit a roadblock: we have proved in [8] that, assuming the continuum hypothesis is true, these Boolean completions will just be instances of the well-known "continuum-collapsing" algebra, which *is* flexibly homogeneous. There may be a more exotic model of set theory in which the continuum hypothesis is false and the Boolean completions of the lattices are not continuum-collapsing algebras, but this seems unlikely to us.[5]

Bergsonian axioms as a robust solution to Russell's paradox

One fascinating suggestion does emerge, despite the flexible-homogeneity issue, from the idea that some continuum-collapsing algebra might be used to generate a Bergson history. (There are other algebras that collapse the continuum besides the well-known one just referenced, and some of these are *not* flexibly homogeneous.) If this suggestion proved workable, it could supply a robust solution to some of the deepest puzzles in the foundations of mathematics. We are referring here to the paradoxes of set theory, notably the version discovered by the aforementioned Russell, which we will briefly lay out. Consider the set of all sets. Being a set itself, it contains itself as a member. Consider next the subset R of this set, defined to have as members all the sets that *don't* have themselves as members. Is R a member of itself? If it is then, by its own definition, it is not, and vice-versa. This may seem a rather facile riddle to those who have not come across it before, but it is impossible to overstate how grave a threat it was considered, in the early 20th century, to the whole edifice of higher mathematics.

The consensus "solution" to this paradox is essentially to command, "Thou shalt not call the collection of all sets a set; thou shalt call it a proper class." This works, in the sense that it keeps contradictions out of mathematics, but it is pretty clearly just a linguistic dodge. Russell's own attempt at a solution, laid out in *Principia Mathematica*, was a heavily embroidered version of this dodge. Bergson's insight, however, promises to truly resolve the paradox. For it lets us see the set of all sets as growing over time: if we consider

[5] For some more intriguing, but even more speculative, connections between physics and our Bergson history setup, see [3].

that the set of all sets at some point in the past is a member, not of itself, but of a larger set of all sets now, we avoid the paradoxes without recourse to an arbitrary and dubious linguistic prohibition.[6] (One has to wonder whether, had Russell taken Bergson's ideas seriously, rather than as target practice for his Oxbridgian condescension, the study of the foundations of mathematics might have evolved in more interesting directions.)

The Russell-type paradoxes are admittedly something of a niche philosophical concern, and the "proper class" dodge has effectively neutered the paradoxes already, from the perspective of working mathematicians. So we would not necessarily expect a more robust solution to occasion widespread celebratory parades. However, the reasoning behind our proposed solution has another benefit: it gives us a firmer understanding of *why* new possibilities should arise over time.

We might put it this way: the Russell-type paradoxes, once a thorn in the side of logic, can be understood as something like the engine of the universe. The universe evolves over time because it is continually trying, and necessarily failing, to form a complete, closed set of possibilities. The attempts fail because considering all sets together *as* a set introduces a new set beyond the ones being considered. But *who* is doing the considering here, and *how* does this new set emerge? If there were no better answer available than "by the mystical mental effort of some sort of Berkeleian God," then this line of thinking might be a philosophical dead end. But the Bergsonian axioms suggest a more philosophically innocuous way that a set-of-all-sets might emerge: as a result of a gradual growth of new mathematical possibilities, which emerge when and only when vanishingly little work is required for them to be derived from previously possible structures.

Conclusion

The robust solution that a Bergsonian perspective can provide for the paradoxes of the foundations of mathematics is, in our view, a compelling reason to study that perspective more deeply. But this reason is certainly secondary to the main motivation, which we would like, by way of concluding, to emphasize.[7] The possible-worlds framework, which has more and more settled into our intellectual life as a default way of seeing the world, tends to deeply demoralize those of us prone to dwelling on its consequences. It robs us of agency; it forecloses any but the most trivial accounts of free will. Bergson's more life-affirming way of seeing the world is far preferable — if it can be reconciled with science. As of today, Bergson's core insight, that life is the continual creation of new possibilities, has not been given a compelling interpretation that would make it fully welcome in scientific circles.

[6] For a deeper analysis of how Bergson's ideas relate to these paradoxes, see [4]. We should perhaps mention that, in order for a newly-created real number to turn an old mathematical universe into a set within a new, larger one, it would have to collapse all the infinite cardinal numbers of the old universe. This is an extremely tall order if your mathematical universe is a model of ZFC (Zermelo-Frankel set theory including the axiom of choice), but if you reject the nonconstructive axioms of choice and power-set and work with the smaller set theory ZF^-, there may be only one infinite cardinal. In this case if would suffice to collapse the continuum, which with forcing is as easy as falling off a log.

[7] It should be noted that these motivations are not truly independent; we have explored in [4] the deep connections between Bergson's concerns and the Russell-type paradoxes.

The Bergsonian axioms project aims to fill this gap. The present author regrets that his mathematical skills have been inadequate to determine whether the axioms are satisfiable, but remains convinced that the question is urgent, and also that infrastructure he has built up for investigating it is a solid one for anyone interested in taking up the task.

References

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