The (hitherto-but-no-longer?) main conjecture of the Bergsonian axioms project is false

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The main conjecture of the “Bergsonian axioms for physics” project has been:

> When $\mathcal{S} \subset \mathcal{R}$ is a proper simple inclusion of injective type III von Neumann factors, $B(\mathbb{P}_\mathcal{S})$, the boolean completion of $\mathcal{S}$'s projection lattice $\mathbb{P}_\mathcal{S}$, embeds naturally as a rigid guaranteed-forcing-distinct complete subalgebra of $B(\mathbb{P}_\mathcal{R})$, defined likewise.

(Relevant definitions are given just below.) We show that this conjecture is false: $B(\mathbb{P}_\mathcal{R})$ turns out, assuming CH (the Continuum Hypothesis), to be the standard boolean algebra for collapsing $\aleph_1$ to $\aleph_0$, which has no rigid proper complete subalgebras at all. Note that the assumption of CH is legitimate within the Bergsonian axioms project, because the model of set theory in which the project defines the algebras is L (the constructible universe), in which CH holds.

Definitions

We assume basic knowledge of complete boolean algebras and subalgebras (as in [2]) and of von Neumann algebras (as in [3]), especially of projection lattices and Murray-von-Neumann-equivalence of projections.

The projection lattice of a von Neumann algebra $\mathcal{R}$ will be denoted $\mathbb{P}_\mathcal{R}$. $\mathbb{P}^+_\mathcal{R}$ means $\mathbb{P}_\mathcal{R} \setminus \{0\}$. When $A, B \in \mathbb{P}$, we write $A \perp B$ to mean they are orthogonal ($AB = 0$).

Partitions of $\mathbb{P}_\mathcal{R}$ will be central to our arguments; by this we always mean a *lattice* partition, i.e. a maximal pairwise-disjoint subset of $\mathbb{P}^+_\mathcal{R}$.

The boolean completion of a separative poset like $\mathbb{P}_\mathcal{R}$ will be denoted $B(\mathbb{P}_\mathcal{R})$.

When $C \subseteq B$ is a boolean subalgebra inclusion, we call it a *rigid (boolean) inclusion*, and call C a *rigid subalgebra* of B, if B has no nontrivial automorphism that leaves C fixed element-wise. We call it a *guaranteed-forcing-distinct* inclusion if all conditions force G (the generic ultrafilter on B) and $C \cap G$ to give distinct generic extensions of the ground model.

Let $\mathcal{S} \subset \mathcal{R}$ be as in the main conjecture, i.e. a proper simple inclusion of injective type III von Neumann factors. Because our disproof of the main conjecture will show that $B(\mathbb{P}_\mathcal{R})$ has no rigid proper complete subalgebras at all, the precise definition of a *simple subfactor* is irrelevant to it; but the definition can be found in [1].

For cardinals $\kappa > \aleph_0$, C_κ is the set of finite sequences of ordinals $< \kappa$, ordered by reverse inclusion; this is the standard poset used to force κ to become countable.

Recall that when A, B are posets, a *dense embedding* $\phi : A \to B$ is an order-isomorphism onto a subset of B such that every $b \in B$ has some $a \in A$ satisfying $\phi(a) \leq b$.

If one separative poset embeds densely into another, then the embedding extends naturally to an isomorphism between their boolean completions. Thus to show $B(C_\kappa)$ isomorphic to $B(\mathbb{P}_\mathcal{R})$, it suffices to show that C_κ embeds densely into $\mathbb{P}^+_\mathcal{R}$.

1 Each nontrivial lattice partition of a type III $\mathbb{P}$ is uncountable

Our argument that $B(\mathbb{P}_\mathcal{R})$ is (isomorphic to) $B(C_{\aleph_1})$ under CH depends on the fact that all nontrivial partitions of $\mathbb{P}_\mathcal{R}$ are uncountable. There may be a simpler proof of this fact than the following one (perhaps a proof that does not need to consider $\mathcal{R}$ as acting on a particular Hilbert space); but this is the best we can do at the moment.

The strategy is to take any countable subset $\{P_n\} \subseteq \mathbb{P}^+ \setminus \{1\}$ and show it is not a lattice partition, by finding $Q \in \mathbb{P}^+$ disjoint from all the P_n. We start by setting $Q_0 = P_0$, and then inductively define each Q_{n+1} by "slightly trimming and rotating" Q_n so as to separate it from P_n by at least a certain angle θ_{n+1}. Q will be the strong-operator-topology limit of these Q_n.

In order to argue geometrically we assume $\mathcal{R}$ to act on a Hilbert space H. The angle between a unit vector $v \in H$ and the range of a projection P is $\arccos(\|Pv\|)$. We define the separation between non-null projections Q and P as the infimum of this angle over all unit vectors in Q's range:

$$sep(Q, P) = \inf\{\arccos(\|Pv\|) : v \in \operatorname{ran}(Q), \|v\| = 1\}.$$

It is easy to check that this can be written equivalently as

$$sep(Q, P) = \arccos(\sup\{|\langle v, w\rangle| : v \in \operatorname{ran}(Q), w \in \operatorname{ran}(P), \|v\| = \|w\| = 1\}).$$

This form makes it clear that $sep(Q, P) = sep(P, Q)$.

Fix a faithful normal state ϕ on $\mathcal{R}$ to keep track of the "size" of the Q_n, so that they do not vanish in the limit.

Lemma 1.1 (Trimming lemma) *Given $P, Q \in \mathbb{P}^+$ and $\epsilon > 0$, there exist $Q_{close}, Q_{far} \in \mathbb{P}$ such that Q has orthogonal decomposition $Q = Q_{close} + Q_{far} + (Q \wedge P)$, and $sep(P, Q_{far}) > 0$, and $\phi(Q_{close}) < \epsilon$.*

This is a consequence of the spectral theory for self-adjoint operators in a von Neumann algebra; see [3], Theorems 5.2.2 and 5.2.3.

Let $Q' = Q - (Q \wedge P)$; it then suffices to find $Q_{close}, Q_{far} \in \mathbb{P}$ as demanded, such that $Q' = Q_{close} + Q_{far}$. Consider the operator $PQ \in \mathcal{R}$. The basic idea will be to take PQ's polar decomposition, and then a sufficiently thin "slice" of the spectral resolution of the self-adjoint part.

For $A \in \mathcal{B}(H)$, the *polar decomposition* of A is a pair U, T of operators such that $A = UT$, with T a non-negative self-adjoint operator, and U a partial isometry whose initial space is the closure of T's range. By [3], Proposition 6.1.3, a polar decomposition $A = UT$ exists, with $U, T \in \mathcal{R}$.

Let UT be a polar decomposition of PQ'. Note that T's right support projection $r(T)$ (whose range is the orthogonal complement of T's kernel) is $\leq Q'$.

Suppose first that $\|T\| < 1$. Then $\|T\|$ is an upper bound < 1 to $\|Pv\|$ for unit $v \in \operatorname{ran}(Q')$; and $\arccos(\|T\|)$ is therefore a lower bound > 0 to $sep(Q', P)$. In this case we simply set $Q_{close} = 0$ and $Q_{far} = Q'$.

If $\|T\| = 1$, consider the spectral resolution of T. The spectral theorem allows us to assert: for all $0 < \lambda < 1$, there exists $A_{(\lambda,1]} \in \mathbb{P}$, $A_{(\lambda,1]} \leq r(T)$, such that for all unit $v \in \operatorname{ran}(Q')$, $v \in \operatorname{ran}(A_{(\lambda,1]}) \iff \|Pv\| \in (\lambda, 1]$. Since Q' is the projection left over when $P \wedge Q$ was subtracted from Q, no unit $v \in \operatorname{ran}(Q')$ satisfies $\|Pv\| = 1$. It follows that as $\lambda \to 1$, the projections $A_{(\lambda,1]}$ converge to the null projection in the strong operator topology. Since ϕ is a normal state it is continuous in the s.o.t., so $\phi(A_{(\lambda,1]})$ will converge to 0. Thus we may choose λ sufficiently close to 1 that $\phi(A_{(\lambda,1]}) < \epsilon$. Let $Q_{close} = A_{(\lambda,1]}$ for this λ and let $Q_{far} = Q' - Q_{close}$. We have $\|Tv\| \leq \lambda$ for all unit $v \in Q_{far}$, so also $\|Pv\| \leq \lambda$ for these v. Thus there is a positive lower bound for $\arccos(\|Pv\|)$ for these v, and

$$sep(Q_{far}, P) = \inf\{\arccos(\|Pv\|) : v \in \operatorname{ran}(Q_{far}), \|v\| = 1\} > 0. \quad \Box$$

Lemma 1.2 *If $Q \perp P$ and $Q \perp R$ then $sep(P + Q, R) = sep(P, R)$.*

This is easily verified. $\Box$

Lemma 1.3 *If A, B, C, D are projections that are mutually orthogonal except that possibly $A \not\perp B$ and possibly $C \not\perp D$, then $sep(A + C, B + D) = \min((sep(A, B), sep(C, D))$.*

Note that the expression $sep(A + C, B + D)$ is well-defined because its arguments are projections, by virtue of the orthogonality relations $A \perp C$ and $B \perp D$. By the second form of the definition of *sep* we have

$$\begin{aligned} sep(A + C, B + D) &= \arccos(\sup\{|\langle a + c, b + d\rangle| : a \in \operatorname{ran}(A)\ etc.; \|a + c\| = \|b + d\| = 1\}) \\ &= \arccos(\sup\{|\langle a, b\rangle + \langle a, d\rangle + \langle c, b\rangle + \langle c, d\rangle| : ...\}) \\ &= \arccos(\sup\{|\langle a, b\rangle + \langle c, d\rangle| : ...\}). \qquad (*) \end{aligned}$$

(For the second line we have used linearity of inner product, and for the third line we have used $A \perp D$ and $C \perp B$.) Now note

$$sep(A, B) = \arccos(\sup\{|\langle a, b\rangle| : a \in \operatorname{ran}(A), b \in \operatorname{ran}(B), \|a\| = \|b\| = 1\}),$$

and that $sep(C, D)$ is defined likewise for $c \in \text{ran}(C), d \in \text{ran}(D)$. It is straightforward to check that if $sep(A, B)$ so defined is $\geq sep(C, D)$ then $(*)$ will equal $sep(A, B)$, and if $sep(C, D)$ is greater then $(*)$ will equal $sep(C, D)$. $\square$

The following lemma will be our tool for separating the Q_n from the P_n.

Lemma 1.4 (Rotating lemma) *Let $P, Z \in \mathbb{P}^+$ and $Q_{in}, Q_{far} \in \mathbb{P}$ satisfy:*
(1) $Q_{in} \leq P$;
(2) $Z \perp P$;
(3) $Q_{in} + Z \perp Q_{far}$;
(4) Q_{in} and Q_{far} are not both null; and
(5) if $Q_{far} \neq 0$ then $sep(P, Q_{far}) > 0$.
Then for all $\epsilon > 0$ there exists $Q_\theta \in \mathbb{P}$ such that
(a) $Q_\theta \perp Q_{far}$,
(b) $\|Q_\theta - Q_{in}\| < \epsilon$,
(c) $sep(P, Q_\theta + Q_{far}) > 0$, and
(d) $|\phi(Q_{far} + Q_\theta) - \phi(Q_{far} + Q_{in})| < \epsilon$.

If $Q_{in} = 0$, let $Q_\theta = 0$, and all the requirements are met for any $\epsilon > 0$ (in particular, (c) follows from (4) and (5)).

Assume then that $Q_{in} \neq 0$. We invoke the Murray-von-Neumann equivalence of all non-null projections in a type III factor to obtain a partial isometry $V \in \mathcal{R}$ whose initial space is Q_{in}'s range (i.e. $V^*V = Q_{in}$) and whose final space is Z's range (i.e. $VV^* = Z$). Suppositions (1) and (2) imply $Q_{in} \perp Z$. Therefore V_θ defined for $\theta \in (0, \pi/2]$ as

$$V_\theta := \cos(\theta) Q_{in} + \sin(\theta) V,$$

is a partial isometry. V_θ can be described as a partial rotation by θ radians of Q_{in}'s range towards Z's range. Let Q_θ be $V_\theta V_\theta^*$, a projection whose range is Q_{in}'s range rotated by θ radians towards Z. The following may be logged as two more established facts:

(6) $sep(Q_{in}, Q_\theta) = \theta$.
(7) $Q_\theta \leq (Q_{in} + Z)$.

Requirement (a) now holds for all θ by (3) and (7).

Requirement (b) holds for sufficiently small θ because $\|Q_\theta - Q_{in}\| \leq \theta$ (by (6) and simple trigonometry).

If either Q_{far} or $(P - Q_{in})$ is null, then requirement (c) follows from Lemma 1.2, because in these cases $sep(P, Q_\theta + Q_{far}) = sep(Q_{in}, Q_\theta)$, which by (6) is > 0. To show this in the $Q_{far} = 0$ case, use the fact that P can be orthogonally decomposed as $P = Q_{in} + (P - Q_{in})$; then we have $sep(P, Q_\theta + Q_{far}) = sep(Q_{in} + (P - Q_{in}), Q_\theta)$, which by Lemma 1.2 and $(P - Q_{in})$'s orthogonality to both Q_{in} and Q_θ, equals $sep(Q_{in}, Q_\theta)$. In the $(P - Q_{in}) = 0$ case, $P = Q_{in}$ by (1), so $sep(P, Q_\theta + Q_{far}) = sep(Q_{in}, Q_\theta + Q_{far})$; by (3) and (7), Q_{far} is orthogonal to both Q_{in} and Q_θ, so again we get $sep(Q_{in}, Q_\theta)$.

In the remaining case $Q_{far} \neq 0$ and $(P - Q_{in}) \neq 0$; here we show requirement (c) using Lemma 1.3. Set $A = Q_{in}$, $B = Q_\theta$, $C = P - Q_{in}$, $D = Q_{far}$. Lemma 1.3's orthogonality requirements are then easily verified from (1), (2), (3) and (7), and we obtain:

$$\begin{aligned} sep(P, Q_\theta + Q_{far}) &= sep(Q_{in} + (P - Q_{in}), Q_\theta + Q_{far}) \\ &= \min(sep(Q_{in}, Q_\theta), sep((P - Q_{in}, Q_{far})) \\ &= \text{ the lesser of } \theta \text{ [by (6)] and a value } > 0 \text{ [by (5)]} \\ &> 0. \end{aligned}$$

Requirement (d), like (b), is met for sufficiently small θ, because Q_θ converges to Q_{in} in norm as $\theta \to 0$. Thus to fulfill all the requirements it suffices that θ be sufficiently close to 0. □

Lemma 1.5 *When $\mathbb{P}$ is the projection lattice of a type III von Neumann factor $\mathcal{R}$, each nontrivial lattice partition of $\mathbb{P}$ is uncountable.*

Let $\{P_n : n < \omega\} \subseteq \mathbb{P}^+ \setminus \{1\}$ be arbitrary. We will find $Q \in \mathbb{P}^+$ that is disjoint from all the P_n. Thus even if $\{P_n\}$ is a pairwise-disjoint subset of $\mathbb{P}^+$, it is not a *maximal* such subset, i.e. not a partition of $\mathbb{P}$. In order to avoid discussing separately the case of finite sets $\{P_n : n < m\}$, we allow the same projection to be indexed by many (possibly infinitely many) n in $\{P_n\}$.

We obtain Q as the strong-operator-topology limit of a sequence $\{Q_n\}$ of projections. We inductively define each Q_{n+1} by "slightly trimming and rotating" Q_n so as to separate it from P_n by at least a certain angle θ_{n+1}.

Set $Q_0 = P_0$ and $\theta_0 = \phi(P_0) \ / \ 2$. We now give the inductive definition of the Q_{n+1} for $n \geq 0$ and of associated angles θ_{n+1}.

First trimming step:

We first make an orthogonal decomposition of Q_n from the previous stage,

$$Q_n = Q_n^{in} + Q_n^{close} + Q_n^{far,untrimmed},$$

where "in," "close," and "far" connote where the ranges of the projections lie relative to P_n's range. We will throw out Q_n^{close}; this is our first "trimming" operation. Invoking Lemma 1.1, we make the above decomposition in such a way that:

— $Q_n^{in}, Q_n^{close}, Q_n^{far,untrimmed} \in \mathbb{P}$,

— $Q_n^{in} = Q_n \wedge P_n$,

— if $Q_n \leq P_n$ then $Q^{far,untrimmed} = Q_n^{close} = 0$, otherwise $sep(Q_n^{far,untrimmed}, P_n) > 0$,

— $\phi(Q_n^{close}) < \theta_n/3$ (note Q_n^{close} is allowed to be 0).

Second trimming step:

The second trimming step removes a small subprojection of $Q_n^{far,untrimmed}$, leaving Q_n^{far}; this is to "make room to rotate Q_n^{in} towards."

Let $Z_n \in \mathbb{P}^+$ be an arbitrary projection $\leq (1 - P_n)$ such that, when Y_n denotes the projection onto the closed range of $Q_n^{far} Z_n$, we have $\phi(Y_n) < \theta_n/3$. Let $Q_n^{far} = Q^{far,untrimmed} - Y_n$ (note this entails $Q_n^{far} \perp Z_n$).

Note that by trimming off Q_n^{close} and Y_n from Q_n, we have obtained a projection $Q_n^{far} + Q_n^{in}$ whose ϕ-value is lower than Q_n's by at most $2\theta_n/3$.

Rotation step:

If $Q_n^{far} > 0$ then let $\epsilon = \min(\theta_n, sep(Q_n^{far}, P_n))/3$; otherwise let $\epsilon = \theta_n/3$. Then obtain Q_θ from Lemma 1.4 using as P, Z, Q_{in}, Q_{far} our corresponding projections subscripted with n, and using the ϵ just specified.

Definition of Q_{n+1}:

Define $Q_{n+1} = Q^{far} + Q_\theta$ using the Q_θ just obtained. Clause (c) of Lemma 1.4 ensures that $sep(P_n, Q_{n+1}) > 0$; let θ_{n+1} be the lesser of $sep(P_n, Q_{n+1})$ and the θ specified by Lemma 1.4. We may then conclude the following from the Lemma:

— (A) $0 < \theta_{n+1} < \theta_n/3$; thus $\theta_n < (\phi(P_0)/2)3^{-n}$;
— (B) $sep(P_n, Q_{n+1}) \geq \theta_{n+1}$;
— (C) Q_{n+1} has orthogonal decomposition $Q_n^{far} + Q_\theta$, such that $Q_n^{far} \leq Q_n$ and no vector in Q_θ's range lies at an angle $> \theta_{n+1}$ from Q_n's range;
— (D) $|\phi(Q_{n+1}) - \phi(Q_n)| < \theta_n/3 + \theta_n/3 + \theta_n/3 = \theta_n$.

Verification that the Q_n converge to Q with desired properties:

The key point, (A), is that the angles θ_n by which Q's approximations are rotated vanish quickly.

(A) and (C) imply the Q_n will converge in the strong operator topology of $\mathcal{B}(H)$ to an operator Q which, since $\mathbb{P}$ is closed in this topology, is in $\mathbb{P}$.

Furthermore (A) and (D) imply that this Q is not null.

Finally, (A), (B), (C) imply that $Q \wedge P_n = 0$ for all n. By (B), the minimal angle separating a vector in P_n's range from a vector in Q_{n+1}'s range is θ_{n+1} radians; vectors in Q_{n+2}'s range are rotated vectors from Q_{n+1}'s range but by (C) the rotation can be no greater than $\theta_{n+1}/3$ radians, and no greater than $\theta_{n+1}/9$ at the next stage, etc. Thus even if a vector in Q_{n+1}'s range at the minimal angle to P_n's range is rotated in successive steps directly back towards P_n's range, the sum of the magnitudes of these rotations cannot exceed $(1/3 + 1/9 + \ldots)\theta_{n+1} = \theta_{n+1}/2$.

Thus Q witnesses the failure of $\{P_n\}$ to be a maximal pairwise-disjoint subset of $\mathbb{P}^+$, i.e. its failure to be a partition of $\mathbb{P}$. □

2 CH implies $B(\mathbb{P}_{\mathcal{R}}) = B(C_{\aleph_1})$

Lemma 2.1 *Suppose that for some faithful normal state ϕ on $\mathcal{R}$, and all $P \in \mathbb{P}^+$, and all $\epsilon > 0$, there exists a lattice partition X of P such that (i) $|X| = 2^{\aleph_0}$, (ii) $Q \in X \Rightarrow$*

$\phi(Q) < \epsilon$, and (iii) for all $P' \in \mathbb{P}^+$, $P' \leq P$, there exists $Q \in X$ such that either $P' \leq Q$ or $Q \leq P'$. Then there is a dense embedding of $C_{2^{\aleph_0}}$ into $\mathbb{P}^+$.

Let $P_{\langle\rangle}$ (indexed by the empty sequence) be $1 \in \mathbb{P}$. Let $\{P_{\langle\alpha\rangle} : \alpha < 2^{\omega_0}\}$ be a lattice partition of $P_{\langle\rangle}$ of the kind supposed, with $\epsilon = 1/2$. Next, for each α obtain a lattice partition $\{P_{\langle\alpha,\beta\rangle} : \beta < 2^{\omega_0}\}$ of $P_{\langle\alpha\rangle}$, with $\epsilon = 1/4$. Continue in this manner so that for every finite sequence $\vec{\alpha}$ of ordinals $< 2^{\omega_0}$, $\phi(P_{\vec{\alpha}}) < 2^{-|\vec{\alpha}|}$.

It is easily verified that for all $n > 0$, $X_n := \{P_{\vec{\alpha}} : |\vec{\alpha}| = n\}$ is a lattice partition of $1 \in \mathbb{P}$ satisfying (i), (ii), (iii) for $\epsilon = 2^{-|\vec{\alpha}|}$.

The $P_{\vec{\alpha}}$ are then dense in $\mathbb{P}^+$: given $P \in \mathbb{P}^+$, fix n such that $2^{-n} < \phi(P)$; then by (iii), some $P_{\vec{\alpha}} \in X_n$ satisfies either $P \leq P_{\vec{\alpha}}$ or $P_{\vec{\alpha}} \leq P$; but the first alternative is excluded because each $P_{\vec{\alpha}} \in X_n$ satisfies $\phi(P_{\vec{\alpha}}) < 2^{-n} < \phi(P)$. □

Theorem 2.2 *Under the Continuum Hypothesis, a separably-acting type III factor's projection lattice $\mathbb{P}$ is forcing-equivalent to the standard $\aleph_1$-collapsing poset $C_{\aleph_1}$.*

Recall that the Continuum Hypothesis means $2^{\aleph_0} = \aleph_1$. We show that under CH there exists a dense embedding of $C_{2^{\aleph_0}}$ into $\mathbb{P}^+$. Fix $P \in \mathbb{P}^+$, $\epsilon > 0$, and a faithful normal state ϕ on $\mathcal{R}$; by Lemma 2.1 it suffices to show there exists a lattice partition X of P such that (i) $|X| = 2^{\aleph_0}$, (ii) $Q \in X \Rightarrow \phi(Q) < \epsilon$, and (iii) for all $P' \in \mathbb{P}^+$, $P' \leq P$, there exists $Q \in X$ such that either $P' \leq Q$ or $Q \leq P'$.

CH allows us to enumerate the projections in $\mathbb{P}^+$ that are $\leq P$ as $\{P_\alpha : \alpha < \omega_1\}$.

We define Q_α by induction on α, for all $\alpha < \omega_1$, with the induction hypothesis that the previously-defined Q_β are pairwise-disjoint. At step α, if $P_\alpha \leq Q_\beta$ for some $\beta < \alpha$, let $Q_\alpha = 0$. Otherwise we observe that $\{Q_\beta \wedge P_\alpha : \beta < \alpha\}$ cannot be a partition of P_α (in the lattice $\mathbb{P}$), by Lemma 1.5 and the fact that α is countable. Thus $\{Q_\beta \wedge P_\alpha : \beta < \alpha\}$ is a pairwise-disjoint but not *maximal* pairwise-disjoint subset of the projections in $\mathbb{P}^+$ below P_α; so there exists in $\mathbb{P}^+$ a projection $< P_\alpha$ that is disjoint from all Q_β, $\beta < \alpha$. Let Q_α be such a projection satisfying $\phi(Q_\alpha) < \epsilon$ (it follows from the normality of ϕ, and the fact that these projections form a downward-closed subset of $\mathbb{P}^+$, which has no minimal projections, that we can require an arbitrarily small ϕ value here).

Let X be the set of all the nonzero Q_α so obtained. Clearly requirements (ii) and (iii) are met, and X is pairwise-disjoint. Moreover X must be a *maximal* pairwise-disjoint subset of the projections in $\mathbb{P}^+$ below P — i.e. a partition of P — because any such projection we might hope to add to X was already enumerated as one of the P_α, and either P_α was $\leq Q_\beta$ for some $\beta < \alpha$ or $P_\alpha \wedge Q_\alpha \neq 0$. Finally, by Lemma 1.5 any nontrivial partition of P is uncountable, so under CH any such partition will satisfy (i). □

3 $B(C_{\aleph_1})$ has no rigidly-included guaranteed-forcing-distinct complete subalgebra.

Theorem 3.1 *$B(C_{\aleph_1})$ has no rigidly-included guaranteed-forcing-distinct complete subalgebra.*

This is a consequence of Corollary 26.10 in Jech [2], which states (although with a different symbol used for C_κ): "Let G be a generic filter on C_κ and let X be a set of ordinals in $V[G]$. Then either $V[X] = V[G]$ or there exists a $V[X]$-generic filter H on C_κ such that $V[X][H] = V[G]$."

We invoke this corollary with $\kappa = \aleph_1$. Since $C_{\aleph_1}$ is dense in its own boolean completion $B(C_{\aleph_1})$, if we define G' as G's upward closure in $B(C_{\aleph_1})$, G' will be a generic ultrafilter and we will have $V[G] = V[G']$.

Now suppose that $D \subseteq B(C_{\aleph_1})$ is a guaranteed-forcing-distinct complete subalgebra. Because the elements of $B(C_{\aleph_1})$ can be enumerated in the ground model as $e_\alpha : \alpha < \lambda$ for some cardinal λ, $G' \cap D$ is interdefinable with a set $X = \{\alpha : e_\alpha \in G' \cap D\}$ of ordinals in $V[G']$ such that $V[G' \cap D] = V[X]$. By our definition of a guaranteed-forcing-distinct inclusion of boolean algebras, all conditions force $V[G' \cap D] \neq V[G']$. Thus all conditions force the other alternative (from the Corollary just cited) to hold: there exists a $V[X]$-generic filter H on $C_{\aleph_1}$ such that $V[X][H] = V[G]$. Equivalently, there exists a $V[G' \cap D]$-generic ultrafilter H on $B(C_{\aleph_1})$ such that $V[G' \cap D][H] = V[G']$. By a basic result on two-step iterations of forcing algebras (see the remarks in [2] following Lemma 16.3), $B(C_{\aleph_1})$ can be written $D * \dot{E}$, where $\dot{E}$ is a canonical D-name for $B(C_{\aleph_1})$. It can be shown that D will be a rigidly included subalgebra of $D * \dot{E}$ in this case if and only if $\|\dot{E} \text{ is rigid}\| = 1$. But in this case $\dot{E}$ is the name for a standard collapse algebra, which is not rigid. So D is not rigidly included. $\Box$

Corollary 3.2 $B(\mathbb{P})$ *has no rigidly-included guaranteed-forcing-distinct complete subalgebra.*

Follows from Theorems 2.2 and 3.1. $\Box$

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