AQFT as a possible source of self-constructing continua

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Abstract

The forcing construction in [6] was meant to produce *self-constructing families of continua*, which formalize the idea that the totality of real numbers might grow "organically," but two problems were identified there that prevent forcing posets from working when used in the construction. We show here how a forcing poset that avoids both problems could, under a conjecture about von Neumann subfactors, be obtained from certain models of AQFT (algebraic quantum field theory). The *simple subfactors* originally identified by R. Longo provide our main tool.

1 Introduction and outline

The two axioms defining a *self-constructing family of continua* (restated below in Section 2.1) formalize the idea that the continuum (the set of all real numbers) could grow organically as more complicated reals are constructed out of less complicated ones. For a full statement of this intuitive idea and an argument that the axioms capture it faithfully, see [6]. The technique suited to produce models of the axioms is set-theoretic forcing, in which a real number's degree of genericity captures how "complicated" it is. Following [6] we will force over the constructible set universe L to obtain candidate models $\mathcal{F}(B, \chi, G)$, where B is a boolean algebra, χ is a function that determines which subalgebras of B contribute continua to the model, and G is a generic ultrafilter on B. (For the precise definition of $\mathcal{F}(B, \chi, G)$ see Section 2.2.) It is currently unknown whether candidates of this form (or any nontrivial form) can satisfy the axioms. [6] showed that they cannot when the best-known forcing algebras are used as B. The present paper is a plan for designing a B suitable for yielding a model, using structures derived from AQFT (algebraic quantum field theory). We begin by explaining why AQFT might be useful towards this end, and by outlining our plan.

The search for a suitable B. By a *suitable* boolean algebra B we will mean one that is free of two properties, identified in [6] and restated below in Section 7, that would (independently) prevent $\mathcal{F}(B, \chi, G)$ from satisfying the self-construction axioms. The *flexible homogeneity* property afflicts the best-known forcing algebras, namely the Cohen-real and random-real algebras. If such a B is used to force a generic real number x, then B will, informally speaking, have too many symmetries (automorphisms) for x to be constructible from an appropriate collection of less-generic reals. It is suggested in [6] that we avoid all symmetries by using a rigid boolean algebra, i.e. a B with no nontrivial automorphisms. But [6]'s attempt to do this yielded a B that suffers from the second problem (the *bounded predecessors* property), which also thwarts self-construction. What is needed is an algebra whose subalgebras are both rigidly and densely nested.

The closest analog of this situation that we have found in the mathematical literature comes from the field of von Neumann algebras, namely, the *simple subfactor inclusions* $\mathcal{R} \subset \mathcal{S}$ first identified by R. Longo in [4]. Such an inclusion is *-algebraically rigid in the sense that no nontrivial *-automorphism of $\mathcal{S}$ leaves each $\mathcal{R}$ member fixed. We will isolate here a Main Conjecture under which the projection lattices of such rigidly-included von Neumann algebras would yield rigidly-included boolean algebras. Whether these boolean algebras would evade the bounded-predecessors problem, however, is not apparent from the main result in [4], which is that a von Neumann factor $\mathcal{S}$ with certain properties will always have a simple subfactor $\mathcal{R}$. By iterating this result we can obtain a nested sequence $R_0 \supset R_1 \supset R_2 \supset \ldots$ of factors, which under our conjecture would yield boolean subalgebras $B_0 \supset B_1 \supset B_2 \supset \ldots$, which in turn would yield, when used in [6]'s construction, continua $\mathbb{R}_0 \supset \mathbb{R}_1 \supset \mathbb{R}_2 \supset \ldots$. But these $\mathbb{R}_n$, like those produced by rigid algebras in [6]'s example, would not be densely ordered by inclusion, and they would succumb to the bounded-predecessors problem for the same reason that example did.

Happily, there is a robust theory of systems of nested von Neumann algebras that are densely ordered by inclusion, namely algebraic quantum field theory (AQFT). The present paper is a plan for transforming certain true AQFT models into models of a variant AQFT axiom ensuring that the inclusions are rigid. We show (again, subject to the Main Conjecture) that the boolean algebras derived from these models would indeed be suitable for use in $\mathcal{F}(B, \chi, G)$.

In outline, our plan to transform systems of von Neumann algebras into systems of suitable boolean algebras will go as follows.

Section 2 restates the self-construction axioms from [6] and the general $\mathcal{F}(B, \chi, G)$ recipe for obtaining candidate models thereof.

Section 3 states two versions of an axiom defining systems of rigidly-included algebras, first the RIST axiom (for "rigidly-included subalgebra theory") defining systems of von Neumann algebras, and second the RIST-B axiom analogously defining systems of boolean algebras.

Section 4 presents one method of transforming certain AQFT models into RIST models; this way, based on the free product technique due to D. Voiculescu, is technically complex, but a potentially simpler method (whose success is not confirmed) is suggested in an appendix, Section 8.

Section 5 gives the procedure for obtaining rigidly-included boolean algebras from (the projection lattices of) rigidly-included von Neumann algebras, and identifies the Main Conjecture need for it to succeed.

Section 6 shows how a RIST model can — under the Main Conjecture — be transformed into a RIST-B model, using the basic procedure in [8].

Section 7 confirms that any boolean algebra in a RIST-B model is suitable for our purposes, i.e. avoids the two problems identified in [6].

Even if our conjectures are correct and our method does yield a suitable boolean algebra B, the task will still remain to find a χ such that $\mathcal{F}(B, \chi, G)$ is a self-constructing family of continua. That task is unlikely to be easy and is beyond the scope of this paper.

2 Self-construction axioms and the $\mathcal{F}(B, \chi, G)$ recipe

The material in this section is restated (verbatim, in places) from [6].

2.1 The self-construction axioms

The idea of a self-constructing family of continua is rooted in the theory of *constructibility*, for which see [2], Chapter 13. In particular it is based on the *relative constructible hierarchy* $L(X)$ relative to an arbitrary set X (see [2], Definition 13.24). Informally $L(X)$ is the collection of those sets that must exist given that X does (and given all the ordinals, and given that the ZF axioms hold). Formally $L(X)$ is the union of the levels $L_\alpha(X)$, where α ranges over the ordinals, and each level is defined in terms of lower levels exactly as in Gödel's original definition of L, except that the base level $L_0(X)$, rather than being defined as the empty set, is now defined as the transitive closure of $\{X\}$.

The *constructive closure* $\mathbb{R}(X)$ of an arbitrary set X of real numbers is, informally, the set of all real numbers that must exist given that X does; formally, we define $\mathbb{R}(X)$ for any set X as the set of all real numbers in $L(X)$.

A *continuum* is a set X of reals that is constructively closed, meaning $\mathbb{R}(X) = X$.

If a continuum X has form $X = \mathbb{R}(x)$ for some real number x, we say X is *singly generated*, and that x *generates* X.

A collection $\mathcal{N}$ of continua (or of sets more generally) is called *directed* (under the inclusion ordering) if for all $X, Y \in \mathcal{N}$, there exists $Z \in \mathcal{N}$ such that $X, Y \subseteq Z$.

A set $\mathcal{N}$ of continua *self-collects into* another continuum X if the following hold:

(i) $X \notin \mathcal{N}$; (X is a new continuum)
(ii) $X, Y \in \mathcal{N} \Rightarrow (\exists Z \in \mathcal{N})(X, Y \subseteq Z)$; ($\mathcal{N}$ is directed)
(iii) $\mathcal{N} \in L(X)$; (nothing beyond X is used to construct X)
(iv) $(\neg\exists x \in X)(\bigcup\mathcal{N} \subseteq \mathbb{R}(x) \subset X)$; ($\mathcal{N}$ is unbounded below X)
(v) $X = \mathbb{R}(\bigcup\mathcal{N})$. ($X$ is the constructive closure of $\mathcal{N}$'s union)

A *self-constructing family* $\mathcal{F}$ of continua is one that satisfies the following two axioms:

Self-Collection: $(\forall X \in L(\mathcal{F}))(X \in \mathcal{F} \iff$ some $\mathcal{N} \subseteq \mathcal{F}$ self-collects into X, and $X = \mathbb{R}(x)$ for some real $x)$.

Foundation: $(\forall X \in \mathcal{F})(\forall x \in X)(\exists Y \in \mathcal{F})$ $(x \in Y \subseteq X$, and $(\forall Z \in \mathcal{F})(Z \subset Y \Rightarrow x \notin Z))$.

2.2 The $\mathcal{F}(B, \chi, G)$ recipe for sets of continua

As in [6], our search for self-constructing families focuses on candidates that are obtained by forcing over L (the constructible universe of sets) and consist entirely of singly-generated continua. All such candidates have the form $\mathcal{F}(B, \chi, G)$ to be defined presently.

B will always mean some complete atomless boolean algebra that exists in L and is countably completely generated (i.e. is a countably-completely-generated subalgebra of itself, as this is defined just below). B^+ denotes B without its least element 0.

$C \subseteq B$ is a (boolean) *subalgebra* of B if C is has the same greatest and least members (0 and 1) as B, and satisfies the boolean axioms with the operations $\wedge$, $\vee$, $\neg$ inherited from B. Such a C is a *complete subalgebra of* B if it is complete, i.e. each of C's subsets has a greatest lower bound and least upper bound in C, *and if furthermore* these bounds are the same whether calculated in C or B. If, in addition, C has a countable subset X such that the smallest complete subalgebra of B that includes X is C itself, we call C a *countably-completely-generated subalgebra of* B.

$\mathsf{CSAs}(B)$ is the set of countably-completely-generated subalgebras of B that exist in L. $\mathsf{ACSAs}(B) \subseteq \mathsf{CSAs}(B)$ is the set of atomless algebras in $\mathsf{CSAs}(B)$.

G is an ultrafilter on B that is generic over L (meaning that G has nonempty intersection with every order-dense subset of B^+ that exists in L).

We now define our general recipe for a family $\mathcal{F}$ of singly-generated continua,

$$\mathcal{F}(B, \chi, G) \equiv \{\mathbb{R}(G \cap C) : C \in \mathsf{CSAs}(B) \text{ and } \chi(C) \in G\},$$

where χ is a "boolean characteristic function" $\chi : \mathsf{CSAs}(B) \to B$ that determines the conditions under which a subalgebra C will contribute a member $\mathbb{R}(G \cap C)$ to the family.

If $D \subset C$ is a proper inclusion of subalgebras in $\mathsf{CSAs}(B)$, the continuum $\mathbb{R}(G \cap D)$ generated by $G \cap D$ need not be a *proper* subset of $\mathbb{R}(G \cap C)$. As shown in [6], the condition under which it is guaranteed (with boolean value 1) to be a proper subset is this:

$$(\neg \exists e \in C^+)(\forall c \in C)(\exists d \in D)(c \wedge e = d \wedge e).$$

If this condition obtains we will call $D \subset C$ a *guaranteed-forcing-distinct inclusion.* We will also apply this terminology when C and D are known to be bounded lattices but not necessarily boolean algebras.

3 RIST, RIST-B, and simple subfactors

AQFT, algebraic quantum field theory, is a mathematically clean and rigorous framework for formalizing (at least some) relativistic quantum systems; its fundamental object is a map from nested regions of spacetime to correspondingly nested von Neumann algebras that represent quantum observables within those regions. We do not assume familiarity with AQFT, but our starting point will be a one-axiom variant of it called RIST, for "rigidly-included subalgebra theory." See [1] for an introduction to AQFT. We will cite [3] as a reference for von Neumann algebras, with which we do assume familiarity.

As its name implies, RIST is most notably distinguished from true AQFT's by the *rigidity* of the inclusions of its von Neumann algebras; we now define this notion, which will be fundamental to us.

Definition of rigid inclusions. If $A \subseteq B$ is an inclusion of sets on which some relations are defined, we call the inclusion *rigid* with respect to those relations if the only automorphism of B that preserves those relations, while leaving all A-members fixed, is the identity map. We will generally specify the structure with respect to which an inclusion is rigid by writing, e.g., *rigid lattice inclusion* or *rigid *-algebra inclusion*.

What our $\mathcal{F}(B, \chi, G)$ recipe needs is a nested system not of von Neumann algebras but of boolean algebras; we intend to provide such a system in the form a "RIST-B" model, a model of an axiom closely analogous to the RIST axiom but phrased in terms of boolean algebras. (In Section 6 we will show, under the assumption of a certain conjecture about subfactor inclusions, how to transform a RIST model into a RIST-B model.)

Definitions used by RIST.

H is a complex Hilbert space of countably infinite dimension.

M is a $(1+n)$-dimensional Minkowski space, $n \geq 0$.

The relation $\leq$ on $M \times M$ is the usual temporal/causal ordering: for points $x = (x_0, \ldots, x_n)$ and $y = (y_0, \ldots, y_n)$ in M, $x \leq y$ holds if and only if $x_0 \leq y_0$ and (in the $n > 0$ cases) $(y_0 - x_0)^2 \leq (y_1 - x_1)^2 + \ldots + (y_n - x_n)^2$.

The *full past cone* of $x \in M$ is $\{y \in M : y \leq x\}$. The *open past cone* of x is the topological interior of its full past cone, or, in relativistic terms, the set of points in that cone that are in the timelike (as opposed to lightlike) past of x. Future cones are defined likewise with the ordering reversed.

$M^\diamond$ is the double cone in M that is the intersection of $(0, \ldots, 0)$'s open future cone with $(1, 0, \ldots, 0)$'s full past cone.

For $x \in M^\diamond$, O_x denotes the intersection of $M^\diamond$ with x's open past cone.

As in AQFT, the fundamental object is a map $O \mapsto \mathcal{R}(O)$ from open bounded regions of M to von Neumann algebras acting on H, *but in RIST the domain of this map is restricted to the double-cone regions O_x of points $x \in M^\diamond$.*

We recall some terminology for properties of a von Neumann algebra $\mathcal{R}$. $\mathcal{R}$ is a *factor* if its center (the set of all $T \in R$ such that T commutes with every operator in $\mathcal{R}$) is trivial (consists of multiples of the identity operator). A factor $\mathcal{R}$ is *type I_n* if it is *-isomorphic to the algebra $M_n(\mathbb{C})$ of $n \times n$ matrices; it is *type III* if all non-null projections $P, Q \in \mathcal{R}$ are Murray-von-Neumann-equivalent, meaning there exists a partial isometry $V \in \mathcal{R}$ such that $V^*V = P$ and $VV^* = Q$. $\mathcal{R}$ is *almost-finite-dimensional* or A.F.D. if it is the closure in the strong operator topology of the union of some sequence $\mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \mathcal{R}_3 \subseteq \ldots$ of subalgebras where each $\mathcal{R}_n$ is a factor of type I_n.

The RIST Axiom: For all $x < y \in M^\diamond$, $\mathcal{R}(O_x) \subset \mathcal{R}(O_y)$ is a rigid *-algebraic inclusion of (distinct) A.F.D. type III von Neumann factors acting on H.

The RIST-B Axiom: For all $x < y \in M^\diamond$, $B_x \subset B_y$ is a rigid complete guaranteed-forcing-distinct boolean inclusion of countably-completely-generated boolean algebras.

Simple subfactors

The only examples of rigidly included von Neumann algebras that we are aware of are *simple subfactors*, which were introduced by R. Longo in [4]. They are based (as is much of AQFT) in the Tomita-Takesaki modular theory, for a full treatment of which see Chapter 9 of [3]. Let M be a von Neumann factor acting on Hilbert space H, with separating and cyclic vector $\xi \in H$; let J be the modular conjugation of M with respect to ξ; let R be a subfactor of M; then R is called a *simple subfactor* of M if the von Neumann algebra $(R \cup JRJ)''$ generated by the union of R and JRJ is the whole algebra $\mathcal{B}(H)$ of bounded operators on H. Note that the choice of ξ relative to which J is defined does not affect whether this condition holds.

4 Models of RIST

The method we present here of obtaining a RIST model works as follows: start with a certain AQFT model defined on a Minkowski space M, consider the von Neumann algebras $\mathcal{A}(O_p)$ that this model assigns to the open-double-cone regions O_p for $p \in M^\diamond$ (as defined in Section 3 above), and obtain each such algebra's infinite free-product with itself, which we denote by $\mathcal{R}(O_p)$, using the free product technique (due mainly to D. Voiculescu) as applied by R. Longo, Y. Tanimoto, and Y. Ueda in [5].[1] We will prove that in our case the set of all $\mathcal{R}(O_p)$'s satsfies the RIST axiom.

A second and potentially simpler way of transforming the algebras $\mathcal{A}(O_p)$, using the method from R. Longo's original paper [4] on simple subfactors, is relegated to an appendix (Section 8) because we have not been able to verify that it succeeds.

We begin with the definitions of the basic AQFT structure from Section 3 of [5].

[This section is not complete.]

5 Boolean completions and embeddings of projection lattices

In this section we define the boolean completion $B_\mathcal{R}$ of a von Neumann algebra $\mathcal{R}$'s projection lattice $\mathbb{P}_\mathcal{R}$; for von Neumann algebra inclusions $\mathcal{R} \subset \mathcal{S}$, we state the conditions under which $B_\mathcal{R}$ completely embeds into $B_\mathcal{S}$ in a natural way; and for rigid *-algebraic inclusions $\mathcal{R} \subset \mathcal{S}$, we state the further condition under which $B_\mathcal{R}$'s image under this natural embedding is rigidly included in $B_\mathcal{S}$. These completions and embeddings will be the main tools in Section 6's plan to derive RIST-B models from RIST models.

[1]The basic idea for method was suggested to the author by R. Longo in e-mail correspondence.

5.1 Projection lattices and their boolean completions

When H is a Hilbert space we write $\mathcal{B}(H)$ for the von Neumann algebra of all bounded linear operators on H. The set $\mathbb{P}(H)$ of all projections,

$$\mathbb{P}(H) \equiv \{P \in \mathcal{B}(H) : P^2 = P^* = P\},$$

is a complete bounded lattice with meet and join operations defined in the conventional way: $P \wedge Q$ is the projection onto the intersection of P's and Q's ranges and $P \vee Q$ is the projection onto the closed linear span of their ranges' union. (In this lattice, 1 is the identity operator and 0 the null operator.) When $\mathcal{R} \subseteq \mathcal{B}(H)$ is a von Neumann algebra, we write $\mathbb{P}_{\mathcal{R}}$ to mean its *projection lattice* $\mathbb{P}(H) \cap \mathcal{R}$; it is a complete bounded sublattice of $\mathbb{P}(H)$, as a consequence of the fact that a von Neumann algebra is closed in the strong operator topology. We write $\mathbb{P}^+_{\mathcal{R}}$ to mean $\mathbb{P}_{\mathcal{R}} \setminus \{0\}$.

Two elements P, Q of a bounded lattice are *disjoint* if $P \wedge Q = 0$; we call an element P disjoint from a set of elements if it is disjoint from each of them individually. Note that two projections P, Q are disjoint if and only if their ranges are disjoint aside from the null vector, and this holds irrespective of the particular von Neumann algebra $\mathcal{R}$ whose projection lattice $\mathbb{P}_{\mathcal{R}}$ we are considering P and Q to be members of.

A *partition* of an element $P \neq 0$ of a bounded lattice is a maximal pairwise-disjoint subset of the nonzero elements $\leq P$.

Definitions of $[P]_{\mathcal{S}}$ ***and the boolean completion*** $B_{\mathcal{S}}$. When $\mathcal{S}$ is a von Neumann algebra and $P \in \mathbb{P}^+_{\mathcal{S}}$, we write $[P]_{\mathcal{S}}$ to mean P's *downward closure in* $\mathbb{P}^+_{\mathcal{S}}$, i.e.

$$[P]_{\mathcal{S}} \equiv \{Q \in \mathbb{P}^+_{\mathcal{S}} : Q \leq P\};$$

and we then define $B_{\mathcal{S}}$ to be the boolean completion of $\mathbb{P}^+_{\mathcal{S}}$ realized as its regular open algebra relative to the topology with basis $\{[P]_{\mathcal{S}} : P \in \mathbb{P}^+_{\mathcal{S}}\}$. For general facts about obtaining boolean completions of posets this way — including the fact that the resulting boolean algebras are always complete — see [2], Theorems 7.13 and 14.10. Concretely, $B_{\mathcal{S}}$ is the set of all subsets $X \subseteq \mathbb{P}^+_{\mathcal{S}}$ that satisfy

$$(\forall Q \in \mathbb{P}^+_{\mathcal{S}})(Q \in X \iff \text{no } Q' \leq Q \text{ in } \mathbb{P}^+_{\mathcal{S}} \text{ is disjoint from } X), \tag{$\dagger$}$$

and the boolean operations on $B_{\mathcal{S}}$ are defined as follows:

$X \wedge Y \equiv X \cap Y$;

$\neg X \equiv \{P \in \mathbb{P}^+_{\mathcal{S}} : P \text{ is disjoint from } X\}$.

$X \vee Y \equiv \neg(\neg X \wedge \neg Y)$.

The boolean ordering $\leq$ is defined as always by $X \leq Y \iff X \wedge Y = X$, and in $B_{\mathcal{S}}$ this is equivalent to $X \subseteq Y$. Note that the mapping $P \mapsto [P]_{\mathcal{S}}$ is not a lattice embedding because $[P \vee Q]_{\mathcal{S}}$ does not equal $[P]_{\mathcal{S}} \vee [Q]_{\mathcal{S}}$ unless $P \leq Q$ or $Q \leq P$. However, $[P \wedge Q]_{\mathcal{S}} = [P]_{\mathcal{S}} \wedge [Q]_{\mathcal{S}}$

always holds, and basis elements $[P]_{\mathcal{S}}$ are dense in $B^+_{\mathcal{S}}$ in the sense relevant to forcing, i.e., for all $X \in B^+_{\mathcal{S}}$ there exists $P \in \mathbb{P}^+_{\mathcal{S}}$ such that $[P]_{\mathcal{S}} \leq X$.

When $\chi \subseteq \mathbb{P}^+_{\mathcal{S}}$, we write $[\chi]_{\mathcal{S}}$ to denote $\{[P]_{\mathcal{S}} : P \in \chi\}$.

Let us register a few more basic facts about $B_{\mathcal{S}}$.

Lemma 5.1 $P \wedge Q = 0 \iff [P]_{\mathcal{R}} \wedge [Q]_{\mathcal{R}} = 0 \iff$ *P is disjoint from* $[Q]_{\mathcal{R}}$. □

Corollary 5.2 *$\chi \subseteq \mathbb{P}^+_{\mathcal{R}}$ is pairwise-disjoint if and only if $[\chi]_{\mathcal{R}}$ is pairwise-disjoint.* □

Definition. A *basic partition* of $X \in B^+_{\mathcal{S}}$ is a partition of X consisting entirely of basis elements of $B^+_{\mathcal{S}}$. For subsets $\Omega \subseteq B^+_{\mathcal{S}}$, we will call $\Gamma \subseteq B^+_{\mathcal{S}}$ a "basic partition of Ω" (retaining the scare-quotes) if Γ is a basic partition of $\bigvee \Omega$ such that $\bigcup \Gamma \subseteq \bigcup \Omega$.

Lemma 5.3 *For every pairwise-disjoint subset $\chi \subseteq \mathbb{P}^+_{\mathcal{S}}$, $[\chi]_{\mathcal{S}}$ is a basic partition of $\bigvee[\chi]_{\mathcal{S}}$; conversely, every $X \in B^+_{\mathcal{S}}$ has a basic partition.*

If $\chi \subseteq \mathbb{P}^+_{\mathcal{S}}$ is pairwise-disjoint then $[\chi]_{\mathcal{S}}$ is too, by Corollary 5.2; and it is a basic fact about complete boolean algebras that any pairwise-disjoint set of nonzero elements is a partition of its join.

For the converse claim, fix $X \in B^+_{\mathcal{S}}$ and let χ be any maximal pairwise-disjoint subset of X; thus by the first claim $[\chi]_{\mathcal{S}}$ is a partition of *some* $B_{\mathcal{S}}$ member $Y \leq X$. If $Y < X$ strictly, there would exist $Q \in X \setminus Y$, and also by (†) a nonzero projection $Q' \in \mathbb{P}^+_{\mathcal{S}}$, $Q' \leq Q$, disjoint from Y. But $Q' \in X$ since X is downward-closed in $\mathbb{P}^+_{\mathcal{S}}$; so Q' contradicts the maximality of χ. □

Lemma 5.4 *For all $\Omega \subseteq B_{\mathcal{S}}$,*

$$\bigvee \Omega = \{Q \in \mathbb{P}^+_{\mathcal{S}} : \textit{ no } Q' \leq Q \textit{ in } \mathbb{P}^+_{\mathcal{S}} \textit{ is disjoint from } \bigcup \Omega\}.$$

Clearly $\bigvee \Omega$ is the least $B_{\mathcal{S}}$ member that contains $\bigcup \Omega$. Suppose Q belongs to the right-hand set in the equation claimed by the lemma; i.e. no $Q' \leq Q$ in $\mathbb{P}^+_{\mathcal{S}}$ is disjoint from $\bigcup \Omega$. Since $\bigcup \Omega \subseteq \bigvee \Omega$, no $Q' \leq Q$ is disjoint from $\bigvee \Omega$ either; thus since $\bigvee \Omega$ satisfies (†) above, $Q \in \bigvee \Omega$.

Conversely, suppose some $Q \in \bigvee \Omega$ did have a $Q' \leq Q$ disjoint from $\bigcup \Omega$. By the definition of $\neg$, we would have $\bigcup \Omega \subseteq \neg[Q']_{\mathcal{S}}$. It would follow that

$$\bigcup \Omega \subseteq \neg[Q']_{\mathcal{S}} \cap \bigvee \Omega = \neg[Q']_{\mathcal{S}} \wedge \bigvee \Omega,$$

so $\neg[Q']_{\mathcal{S}} \wedge \bigvee \Omega$ would contradict the leastness of $\bigvee \Omega$ for containing $\bigcup \Omega$. □

Lemma 5.5 *Every $\Omega \subseteq B^+_{\mathcal{S}}$ has a "basic partition."*

Fix Ω and let Γ be a maximal pairwise-disjoint set of basis elements of $B^+_{\mathcal{S}}$ such that $\bigcup \Gamma \subseteq \bigcup \Omega$. Suppose towards a contradiction that Γ is not a "basic partition of Ω", which can only happen if $\bigvee \Gamma < \bigvee \Omega$ strictly. Then there exists $Q \in \bigvee \Omega$ that is disjoint from $\bigvee \Gamma$ and hence from every member of Γ. Now $Q \in \bigvee \Omega$ entails by Lemma 5.4 that no $Q' < Q$ in $\mathbb{P}^+_{\mathcal{S}}$ is disjoint from $\bigcup \Omega$. Thus there exists $Q' < Q$ such that $Q' \in \bigcup \Omega$. But Q' is disjoint from each Γ member, so $[Q']_{\mathcal{S}}$ violates Γ's supposed maximality. □

5.2 Embeddings of projection lattices' boolean completions

We have mentioned that the inclusion map from $\mathbb{P}_{\mathcal{R}}$ into $\mathbb{P}_{\mathcal{S}}$ is a complete bounded-lattice embedding. Our question now is whether the map $[P]_{\mathcal{R}} \mapsto [P]_{\mathcal{S}}$, defined for $P \in \mathbb{P}^{+}_{\mathcal{R}}$, extends to a complete boolean embedding of $B_{\mathcal{R}}$ into $B_{\mathcal{S}}$. We begin by recalling precisely what is meant by this terminology.

A *bounded-lattice embedding* ϕ of one bounded lattice A into another B is an injective homomorphism, i.e. an injective mapping $\phi : A \to B$ such that for all $x, y \in A$,

$$\phi(1) = 1; \tag{1}$$
$$\phi(0) = 0; \tag{2}$$
$$\phi(x \vee y) = \phi(x) \vee \phi(y); \tag{3}$$
$$\phi(x \wedge y) = \phi(x) \wedge \phi(y). \tag{4}$$

If A and B are also boolean algebras, and for all $x \in A$ we have

$$\phi(\neg x) = \neg\phi(x), \tag{5}$$

then we call ϕ a *boolean embedding.* If moreover A and B are complete and we have the generalizations of (3) and (4) for all subsets $\Omega \subseteq A$, i.e.

$$(a)\ \ \phi(\bigvee \Omega) = \bigvee_{x\in\Omega} \phi(x), \qquad (b)\ \ \phi(\bigwedge \Omega) = \bigwedge_{x\in\Omega} \phi(x), \tag{6}$$

then we call ϕ a *complete boolean embedding.*

We now define the *canonical mapping* $\phi : B_{\mathcal{R}} \to B_{\mathcal{S}}$ by

$$\phi : X \mapsto \bigwedge\{Y \in B_{\mathcal{S}} : X \subseteq Y\}.$$

Lemma 5.6 *If $[P]_{\mathcal{R}} \mapsto [P]_{\mathcal{S}}$ extends to a complete boolean embedding $B_{\mathcal{R}} \to B_{\mathcal{S}}$, then the canonical mapping ϕ is the unique such embedding.*

Let $\gamma : B_{\mathcal{R}} \to B_{\mathcal{S}}$ be an extending embedding as required, and let $X \in B_{\mathcal{R}}$ be arbitrary. Since by hypothesis $\gamma([P]_{\mathcal{R}}) = [P]_{\mathcal{S}} \ni P$ for all $P \in X$, we must have $X \subseteq \gamma(X)$. If $\gamma(X)$ were not the least $B_{\mathcal{S}}$-member that includes X — i.e., $\phi(X)$ — then we would have

$$\gamma(X) = \gamma(\bigvee_{P\in X} [P]_{\mathcal{R}}) > \bigvee_{P\in X} \gamma([P]_{\mathcal{R}}),$$

which would violate clause (6)(a) of the definition of complete boolean embedding. Thus $\gamma(X) = \phi(X)$, and as X was arbitrary, $\gamma = \phi$. □

Lemma 5.7 *For all $X \in B_{\mathcal{R}}$, $\phi(X) \cap \mathbb{P}_{\mathcal{R}} = X$.*

It is immediate from ϕ's definition that $X \subseteq \phi(X) \cap \mathbb{P}_{\mathcal{R}}$, so it suffices to show that if $Q \in \phi(X) \cap \mathbb{P}_{\mathcal{R}}$, then $Q \in X$. Suppose Q violated this, so that by $Q \notin X$ and (†), there exists $Q' \in \mathbb{P}^+_{\mathcal{R}}$, $Q' \leq Q$, such that Q' is disjoint from X. Consider the $B_{\mathcal{S}}$ element $\phi(X) \wedge \neg[Q']_{\mathcal{S}}$, which is strictly $< \phi(X)$. By the definition of $\neg$, $\neg[Q']_{\mathcal{S}}$ is the set of all $P \in \mathbb{P}^+_{\mathcal{S}}$ disjoint from Q'. Thus $X \subseteq \neg[Q']_{\mathcal{S}}$; and we have noted $X \subseteq \phi(X)$, so

$$X \subseteq \phi(X) \wedge \neg[Q']_{\mathcal{S}},$$

contradicting $\phi(X)$'s leastness in $B_{\mathcal{S}}$ for including X (its defining property). □

Corollary 5.8 *ϕ is an injective mapping.* □

Definition of The Complete Embedding Condition for $\mathcal{R} \subseteq \mathcal{S}$**:** For all $P \in \mathbb{P}^+_{\mathcal{R}}$, every partition of P in $\mathbb{P}^+_{\mathcal{R}}$ is also a partition of P in $\mathbb{P}^+_{\mathcal{S}}$.

Lemma 5.9 *If $\mathcal{R} \subseteq \mathcal{S}$ are von Neumann algebras, then $\phi : B_{\mathcal{R}} \to B_{\mathcal{S}}$ is a complete boolean embedding if and only if $\mathcal{R} \subseteq \mathcal{S}$ meets the Complete Embedding Condition.*

For "only if," suppose the Complete Embedding Condition does not hold, as witnessed by $P \in \mathbb{P}^+_{\mathcal{R}}$ and a partition χ of P in $\mathbb{P}^+_{\mathcal{R}}$ which is not a partition of P in $\mathbb{P}^+_{\mathcal{S}}$. This can only be because χ fails to be maximal, as a set of pairwise-disjoint projections $\leq P$ in $\mathbb{P}^+_{\mathcal{S}}$. Let $Q < P$ witness this failure of maximality. Since Q is disjoint from χ,

$$[Q]_{\mathcal{S}} \not\leq \bigvee_{P' \in \chi} [P']_{\mathcal{S}} = \bigvee_{P' \in \chi} \phi([P']_{\mathcal{R}}),$$

but

$$[Q]_{\mathcal{S}} < [P]_{\mathcal{S}} = \phi([P]_{\mathcal{R}}) = \phi(\bigvee [\chi]_{\mathcal{R}}),$$

implying $\phi(\bigvee[\chi]_{\mathcal{R}}) \neq \bigvee_{P' \in \chi} \phi([P']_{\mathcal{R}})$, which violates clause (6)(a) of complete embeddings.

For the "if" direction, let $\mathcal{R} \subseteq \mathcal{S}$ meet the Complete Embedding Condition. We have seen (Corollary 5.8) that ϕ is injective; we must verify that conditions (1)-(6) given above always hold for ϕ. (1) and (2) are trivial. (3) and (4) are special cases of (6)(a) and (6)(b) respectively, so we need not prove them separately.

Let us show that (6)(a) holds. Let $\Omega \subseteq B_{\mathcal{R}}$ be arbitrary. We observe first that $\bigvee_{X \in \Omega} \phi(X)$ is the least $B_{\mathcal{S}}$ member containing $\bigcup \Omega$, and that $\phi(\bigvee \Omega)$ is the least $B_{\mathcal{S}}$ member containing the least $B_{\mathcal{R}}$ member containing $\bigcup \Omega$. Thus we have

$$\bigvee_{X \in \Omega} \phi(X) \leq \phi(\bigvee \Omega).$$

Suppose now that Ω is a counterexample to (6)(a); the above inequality must then be strict:

$$\bigvee_{X \in \Omega} \phi(X) < \phi(\bigvee \Omega).$$

By Lemma 5.5 there exists a "basic partition" Γ of Ω. By the definition of "basic partition" we have $\bigvee \Gamma = \bigvee \Omega$ and so $\phi(\bigvee \Gamma) = \phi(\bigvee \Omega)$. Also by that definition we have $\bigcup \Gamma \subseteq \bigcup \Omega$; and since $\bigvee_{X\in\Gamma} \phi(X)$ is the least $B_{\mathcal{S}}$ member containing $\bigcup \Gamma$, we have

$$\bigvee_{X\in\Gamma} \phi(X) \leq \bigvee_{X\in\Omega} \phi(X).$$

All this is to say that, under the supposition that there exists a counterexample to (6)(a), we have a pairwise-disjoint set Γ of basis elements of $B_{\mathcal{R}}$ such that

$$\bigvee_{X\in\Gamma} \phi(X) < \phi(\bigvee \Gamma). \tag{$*$}$$

Next, we show there must exist some $P \in \mathbb{P}^+_{\mathcal{R}}$ such that $P \in \phi(\bigvee \Gamma) \setminus \bigvee_{X\in\Gamma} \phi(X)$. Suppose not, which by $(*)$ implies that $\bigvee_{X\in\Gamma} \phi(X)$ and $\phi(\bigvee \Gamma)$ have the same intersection with $\mathbb{P}^+_{\mathcal{R}}$. By Lemma 5.7, $\phi(\bigvee \Gamma) \cap \mathbb{P}^+_{\mathcal{R}} = \bigvee \Gamma$. Thus $\bigvee_{X\in\Gamma} \phi(X)$ would include $\bigvee \Gamma$; but $\phi(\bigvee \Gamma)$ is by definition the least $B_{\mathcal{S}}$ element that includes $\bigvee \Gamma$, so we would have $\bigvee_{X\in\Gamma} \phi(X) \geq \phi(\bigvee \Gamma)$, violating $(*)$.

Now fix a $P \in \mathbb{P}^+_{\mathcal{R}}$ as in the previous paragraph and define $\chi \equiv \{Q \wedge P : [Q]_{\mathcal{R}} \in \Gamma\}$. It is easily verified that χ is a partition of P in $\mathbb{P}_{\mathcal{R}}$. By the Complete Embedding Condition, χ is also a partition of P in $\mathbb{P}_{\mathcal{S}}$. Thus every $Q \in \mathbb{P}^+_{\mathcal{S}}$ that is $\leq P$ meets some member of χ, and so fails to be disjoint from $\bigcup \Gamma$. Since $\bigcup \Gamma \subseteq \bigvee_{X\in\Gamma} \phi(X)$, every such Q would fail to be disjoint from $\bigvee_{X\in\Gamma} \phi(X)$; so by (†), $P \in \bigvee_{X\in\Gamma} \phi(X)$, contradicting our choice of P.

Thus we have shown that (6)(a) has no counterexamples. Our next task is to show that (5) doesn't either, i.e. that for arbitrary $X \in B_{\mathcal{R}}$ we have $\neg\phi(X) = \phi(\neg X)$. Observe that $\neg\phi(X)$ is the unique $B_{\mathcal{S}}$ member whose join with $\phi(X)$ is 1 and whose meet with $\phi(X)$ is 0; thus our task is equivalent to proving that $\phi(\neg X)$ meets these two criteria. Plugging in $\{X, \neg X\}$ for Ω in (6)(a) we have:

$$\phi(\bigvee \{X, \neg X\}) = \phi(X) \vee \phi(\neg X).$$

The left-hand side of this equation is $\phi(1)$, i.e. 1, so the join requirement is met. For the meet requirement, suppose it failed, so that $\phi(\neg X) \wedge \phi(X) > 0$. Let Q be any projection in $\phi(\neg X) \wedge \phi(X)$. Q cannot be disjoint from X, lest $\phi(X) \wedge \neg[Q]_{\mathcal{S}}$ be a $B_{\mathcal{S}}$ member that contains X but is smaller than $\phi(X)$, contradicting the latter's leastness for this property. Thus there exists $Q' \leq Q$, $Q' \in \mathbb{P}^+_{\mathcal{S}}$, such that $Q' \in \phi(\neg X)$ and $Q' \leq$ some projection in X. Now by the definition of negation in $B_{\mathcal{R}}$, $\neg X$ is the set of $\mathbb{P}^+_{\mathcal{R}}$ members disjoint from X, and $\phi(\neg X)$ is the least $B_{\mathcal{S}}$ member containing it. But our Q' contradicts this: since supposedly $Q' \in \phi(\neg X)$, $\phi(\neg X) \wedge \neg[Q']_{\mathcal{S}}$ would be a strictly smaller $B_{\mathcal{S}}$ element, and it would still contain $\neg X$ because Q' is $\leq$ some projection in X.

To confirm that (6)(b) holds, note that the $\neg$ operation is an order-reversing involution (in both $B_{\mathcal{R}}$ and $B_{\mathcal{S}}$), and ϕ preserves $\neg$, so if Ω were a counterexample to (6)(b), then $\{\neg X : X \in \Omega\}$ would be a counterexample to (6)(a); but we have shown that no such counterexample exists. □

5.3 The rigidity condition for our boolean embeddings

Let $B_{\mathcal{R}\subseteq\mathcal{S}}$ denote $B_{\mathcal{R}}$'s image under ϕ. Assuming the Complete Embedding Condition holds and that $\mathcal{R} \subseteq \mathcal{S}$ is a rigid *-algebraic inclusion of type III factors, the following condition determines whether $B_{\mathcal{R}\subseteq\mathcal{S}}$ will be a rigid boolean subalgebra of $B_{\mathcal{S}}$; we will prove this in a sequence of lemmas that culminates in Lemma 5.14.

The Rigidity Condition: If there is a nontrivial boolean automorphism of $B_{\mathcal{S}}$ that leaves each $B_{\mathcal{R}\subseteq\mathcal{S}}$-member fixed then there is a nontrivial lattice automorphism of $\mathbb{P}_{\mathcal{S}}$ that leaves each $\mathbb{P}_{\mathcal{R}}$-member fixed.

Lemma 5.10 *If $\mathcal{R} \subseteq \mathcal{S}$ is a rigid *-algebraic inclusion of type III von Neumann factors for which the Complete Embedding Condition holds, $\mathbb{P}_{\mathcal{R}} \subseteq \mathbb{P}_{\mathcal{S}}$ is a rigid lattice inclusion.*

We will use several sub-lemmas to prove this; it is a consequence of results in [9]. That paper makes essential use of the *-algebra $LS(M) \supseteq M$ of *locally measurable operators* relative to a von Neumann algebra M. Because our factors are type III, and because, as noted on p. 5 of [9], $LS(M) = M$ for such factors, we will not bother defining $LS(M)$ here. We will also adopt from [9] the notation $l(X)$ to mean the left support projection of a Hilbert-space operator X, i.e. the projection onto the closure of X's range. Note also that by a "real *-automorphism" ψ of a von Neumann algebra is meant a ring automorphism that preserves the adjoint relation (i.e. $\psi(T^*) = \psi(T)^*$) and is real-linear (i.e. $\psi(rT) = r\psi(T)$ for real r).

We start with the following immediate corollary of Theorems A and B from [9]:

Fact 5.11 (Corollary of [9], Theorems A and B) *Suppose that M is a type III von Neumann algebra, and that $\Phi : \mathbb{P}_M \to \mathbb{P}_M$ is a lattice automorphism; then there exist a real *-automorphism ψ of M and an invertible element $Y \in LS(M)$ such that for all $X \in LS(M)$, $\Phi(l(X)) = l(Y\psi(X)Y^{-1})$.* $\Box$

The next lemma applies Fact 5.11 to our special case, where M is a factor.

Lemma 5.12 *Suppose that M is a type III von Neumann factor, and that $\Phi : \mathbb{P}_M \to \mathbb{P}_M$ is a lattice automorphism; then there exist a (complex-linear) *-automorphism ψ of M and an invertible element $Y \in M$ such that for all $P \in \mathbb{P}_M$, $\Phi(P) = l(Y\psi(P)Y^{-1})$.*

Note first that "$LS(M)$" can be replaced in Fact 5.11 with "M" because we are assuming M is a factor (see p.5 of [9]). Second, we consider only projections $P \in LS(M)$ rather than arbitrary operators $X \in LS(M)$, which allows us to replace the equation $\Phi(l(X)) = l(Y\psi(X)Y^{-1})$ with $\Phi(P) = l(Y\psi(P)Y^{-1})$.

Finally, we justify the replacement of "real *-automorphism" with "(complex-linear) *-automorphism" as follows. Clause (4) of [9]'s Lemma 2.1 entails that when M is a factor, so that 0 and 1 are its only central projections, a real *-automorphism ψ of M must be either a (complex-linear) *-automorphism or a conjugate-linear *-automorphism. Suppose we have an automorphism ψ of the latter type, which along with an operator Y satisfies the lemma's statement. Then the mapping ψ^* defined by $\psi^*(T) \equiv \psi(T^*)$ is an automorphism of M of the former type, since $\psi^*(cT) = \psi((cT)^*) = \psi(\overline{c}T) = c\psi(T)$, where c is a complex number and $\overline{c}$ its complex conjugate. And $\Phi(P) = Y\psi^*(P)Y^{-1}$ still holds for all projections P since $P^* = P$. □

Lemma 5.13 *If $A \in \mathcal{B}(H)$ is invertible and $(\forall P \in \mathbb{P}_{\mathcal{R}})(P = l(APA^{-1}))$, then $A \in \mathcal{R}'$.*

Fix $P \in \mathbb{P}_{\mathcal{R}}$. Since A is invertible we have range(PA) = range(PA^{-1}) = range(P). This along with the lemma's hypothesis ensures that APA^{-1} maps P's range into itself, and (applying the hypothesis to $(1-P)$) that $A(1-P)A^{-1}$ maps $(1-P)$'s range into itself. Since an operator A commutes with a projection P if and only if A maps P's range into itself, and $(1-P)$'s range into itself, A commutes with P. As P was arbitrary, A commutes with all projections in $\mathcal{R}$. It then follows from the way von Neumann algebras are generated by their projections (i.e. the spectral theorem; see [3], Theorem 5.2.2) that A commutes with all of $\mathcal{R}$, i.e. $A \in \mathcal{R}'$. □

Proof of Lemma 5.10.

The lemma's statement supposes that $\mathcal{R} \subset \mathcal{S}$ is a rigid *-algebra inclusion of type III factors. Suppose towards a contradiction of its claim that there exists a nontrivial lattice automorphism Φ of $\mathbb{P}_{\mathcal{S}}$ that acts trivially on $\mathbb{P}_{\mathcal{R}}$. Then by Lemma 5.12 there exist a complex-linear *-automorphism ψ of $\mathcal{S}$ and an invertible $Y \in \mathcal{S}$ such that for all $P \in \mathbb{P}_{\mathcal{S}}$, $\Phi(P) = l(Y\psi(P)Y^{-1})$.

Since Φ supposedly fixes each $\mathbb{P}_{\mathcal{R}}$-member, we have

$$(\forall P \in \mathbb{P}_{\mathcal{R}})(P = l(Y\psi(P)Y^{-1})).$$

By polar decomposition ([3], Prop. 6.1.3), $Y = TU$ for some (unique) self-adjoint $T \in \mathcal{S}$ and unitary $U \in \mathcal{S}$. (In general the U component of a polar decomposition is a partial isometry but for invertible Y, U must furthermore be unitary, and T must be invertible.)

By the unitary implementation theorem ([3], Theorem 7.2.9), there exists a unitary operator $V \in \mathcal{B}(H)$ such that $(\forall T \in \mathcal{S})(\psi(T) = VTV^{-1})$. Thus for all $P \in \mathbb{P}_{\mathcal{R}}$,

$$P = l(TUVPV^{-1}U^{-1}T^{-1}).$$

Invoking Lemma 5.13 with A set to (TUV), we have $(TUV) \in \mathcal{R}'$. Both components of (TUV)'s polar decomposition must also belong to $\mathcal{R}'$ (since $\mathcal{R}'$ is a von Neumann algebra), and these components are just T and UV. So $T \in \mathcal{R}'$ and $UV \in \mathcal{R}'$. Since also $T \in \mathcal{S}$, and $\mathcal{R}$ (being a simple subfactor of $\mathcal{S}$) has trivial relative commutant in $\mathcal{S}$, T must be trivial, i.e. a scalar multiple of 1.

Now since the unitary V has been defined from the *-automorphism of $\mathcal{S}$ that it induces, and the unitary U belongs to $\mathcal{S}$, UV is unitary and induces a *-automorphism of $\mathcal{S}$. This

*-automorphism cannot act trivially on $\mathcal{S}$; otherwise, since T is trivial (of form $c1$), the lattice automorphism

$$\Phi : P \mapsto l(TUVPV^{-1}U^{-1}T^{-1})$$

of $\mathbb{P}_{\mathcal{S}}$ would be trivial, violating our supposition on Φ.

Thus UV induces a nontrivial *-automorphism of $\mathcal{S}$; but since $UV \in \mathcal{R}'$ it acts trivially on $\mathcal{R}$, contradicting the *-algebraic rigidity of $\mathcal{R} \subseteq \mathcal{S}$. $\square$

Lemma 5.14 *If $\mathcal{R} \subseteq \mathcal{S}$ is a rigid *-algebraic inclusion of type III von Neumann factors for which the Complete Embedding Condition holds, then $B_{\mathcal{R}\subseteq\mathcal{S}} \subseteq B_{\mathcal{S}}$ is a rigid boolean inclusion if and only if the Rigidity Condition holds.*

It is immediate from the statement of the Rigidity Condition that if it fails, there is a nontrivial boolean automorphism of $B_{\mathcal{S}}$ that leaves each $B_{\mathcal{R}\subseteq\mathcal{S}}$-member fixed, so the inclusion is not rigid. Conversely, if the Rigidity Condition does hold, there can be no such automorphism, since that would entail the existence of a nontrivial lattice automorphism of $\mathbb{P}_{\mathcal{S}}$ leaving every $\mathbb{P}_{\mathcal{R}}$-member fixed, and by Lemma 5.10 there is no such thing. $\square$

6 The RIST-to-RIST-B recipe

In this section we propose a way of deriving RIST-B models from RIST models, and isolate the conjectures on which its success depends. Our main tools will be the those just defined in Section 5: the boolean completions $B_{\mathcal{R}}$ and $B_{\mathcal{S}}$ and the mapping $\phi : B_{\mathcal{R}} \to B_{\mathcal{S}}$ whose image we denoted $B_{\mathcal{R}\subseteq\mathcal{S}}$.

The RIST-to-RIST-B conjecture: Every inclusion $\mathcal{R}_x \subset \mathcal{R}_y$ of distinct members of a RIST model satisfies the Complete Embedding Condition and the Rigidity Condition (see Section 5), and $\mathbb{P}_{\mathcal{R}_x} \subset \mathbb{P}_{\mathcal{R}_y}$ satisfies the Guaranteed-Forcing-Distinctness Condition (see Section 2.2).

The RIST-to-RIST-B recipe: Relative to a given RIST model $\{\mathcal{R}_x : x \in M^\diamond\}$, let B_x denote, for $x \in M^\diamond$, the algebra $B_{\mathcal{R}_x\subseteq\mathcal{R}_{(1,0,\ldots,0)}}$; we then define the *system of boolean algebras derived from* $\{\mathcal{R}_x : x \in M^\diamond\}$ to be $\{B_x : x \in M^\diamond\}$.

Lemma 6.1 *Suppose the RIST-to-RIST-B conjecture holds; if $\{\mathcal{R}_x : x \in M^\diamond\}$ is a RIST model, and $x \leq y \leq (1,0,\ldots,0)$ are points in $M^\diamond$, and ϕ is the canonical mapping $B_{\mathcal{R}_x} \to B_{\mathcal{R}_y}$ defined above in Section 5, and ϕ' is the canonical mapping $B_{\mathcal{R}_y} \to B_{\mathcal{R}_{(1,0,\ldots,0)}}$; then ϕ and ϕ' are both complete boolean embeddings, and $\phi'[\phi[B_{\mathcal{R}_x}]] = B_x$ (where B_x is as defined in the RIST-to-RIST-B recipe), and B_x is a complete rigid subalgebra of B_y.*

The fact that ϕ and ϕ' are complete boolean embeddings follows from the RIST-to-RIST-B conjecture and Lemma 5.9. Because ϕ is a complete boolean embedding, $\phi[B_{\mathcal{R}_x}]$ (which we have earlier denoted $B_{\mathcal{R}_x\subseteq\mathcal{R}_y}$) is a complete subalgebra of $B_{\mathcal{R}_y}$. It also follows from the conjecture and Lemma 5.14 that this inclusion is rigid. The completeness and rigidity of this inclusion are preserved by ϕ' because ϕ' is a boolean isomorphism of $B_{\mathcal{R}_y}$ onto B_y (as B_y is defined in the RIST-to-RIST-B recipe); i.e. the image of $\phi[B_{\mathcal{R}_x}]$ under ϕ' is a complete rigid subalgebra of B_y. Finally, because a composition of two complete boolean embeddings is a complete boolean embedding, and $\phi' \circ \phi$ extends the mapping $[P]_{\mathcal{R}_x} \to [P]_{\mathcal{R}_{(1,0,\ldots,0)}}$, the uniqueness shown in Lemma 5.6 ensures that $\phi' \circ \phi$ is the canonical mapping of $B_{\mathcal{R}_x}$ into $B_{\mathcal{R}_{(1,0,\ldots,0)}}$, so $\phi'[\phi[B_{\mathcal{R}_x}]] = B_x$. □

Theorem 6.2 *If the RIST-to-RIST-B conjecture holds and $\{\mathcal{R}_x : x \in M^\diamond\}$ is a RIST model, then the system $\{B_x : x \in M^\diamond\}$ of boolean algebras derived from it is a RIST-B model.*

Fix an arbitrary RIST model and arbitrary $x < y \in M^\diamond$; we must show that $B_x \subseteq B_y$, defined as in the RIST-to-RIST-B recipe above, satisfies the requirements of the RIST-B axiom. By Lemma 6.1, B_x is a rigid complete subalgebra of B_y.

It is shown in [7] that forcing with the projection lattice of an A.F.D. non-type-I factor yields a generic filter that is interdefinable with a real number. Since $\mathbb{P}^+_{\mathcal{R}_x}$ has a natural dense embedding into $B^+_{\mathcal{R}_x}$, a generic filter on one induces a generic filter on the other; thus a generic ultrafilter on $B_{\mathcal{R}_x}$ is interdefinable with a real number r, and a boolean algebra whose generic filters are always interdefinable with reals is countably completely generated.

The RIST-to-RIST-B Conjecture explicitly states that $\mathbb{P}_{\mathcal{R}_x} \subset \mathbb{P}_{\mathcal{R}_y}$ satisfies the Guaranteed-Forcing-Distinctness Condition. By the natural dense embeddings $\mathbb{P}_{\mathcal{R}_x} \to B_{\mathcal{R}_x}$ and $\mathbb{P}_{\mathcal{R}_y} \to B_{\mathcal{R}_y}$, and by the natural complete embedding $\phi : B_{\mathcal{R}_x} \to B_{\mathcal{R}_y}$, $B_{\mathcal{R}_x \subseteq \mathcal{R}_y}$ is a guaranteed-forcing-distinct subalgebra of $B_{\mathcal{R}_y}$. Since the natural complete embedding of the latter into $B_{\mathcal{R}_{(1,0,\ldots,0)}}$ is a boolean isomorphism, B_x is a guaranteed-forcing-distinct subalgebra of B_y.

Thus the RIST-to-RIST-B Conjecture implies that $B_x \subseteq B_y$ satisfies all requirements of the RIST-B axiom. □

7 Boolean algebras in RIST-B models are suitable for $\mathcal{F}(B, \chi, G)$

We show that every member algebra B_x of a RIST-B model avoids the two problems identified in [6] that would, if either obtained, prevent $\mathcal{F}(B, \chi, G)$ from satisfying the Self-Collection Axiom (regardless of χ and G). Let us affirm first of all that any member B_x of a RIST-B model qualifies for use in the $\mathcal{F}(B, \chi, G)$ recipe, since it is immediate from the RIST-B axiom that B_x is an atomless, complete, countably-completely-generated boolean algebra.

7.1 The bounded-predecessors problem

We recall the $\mathcal{F}(B, \chi, G)$ recipe from Section 2. If $\chi(C) \in G$ then by definition $\mathbb{R}(G \cap C) \in \mathcal{F}(B, \chi, G)$, and we define the set of $\mathbb{R}(G \cap C)$'s *predecessors in* $\mathcal{F}(B, \chi, G)$ to be

$$\{X \in \mathcal{F}(B, \chi, G) : X \subset \mathbb{R}(G \cap C)\}.$$

We say the pair (B, χ) has the *bounded-predecessors property* if there exist $b \in B^+$ and $C \in \mathsf{ACSAs}(B)$ such that $b \leq \chi(C)$ and b forces $\mathbb{R}(G \cap C)$ to have either a greatest predecessor (under the inclusion ordering) or no predecessors. We say B itself has the bounded-predecessors property, if the pair (B, χ) has it for every χ that maps some $C \in \mathsf{ACSAs}(B)$ to some $b > 0$.

If B has the bounded-predecessors property then, for any nontrivial χ (i.e. any χ that maps some $C \in \mathsf{ACSAs}(B)$ to some $b > 0$), each $b \in B^+$ that witnesses the bounded-predecessors property for (B, χ) forces the Self-Collection axiom to be violated. Proof: $b \leq \chi(C)$ by definition means that b forces $\mathbb{R}(G \cap C) \in \mathcal{F}(B, \chi, G)$, and since C belongs to $\mathsf{ACSAs}(B)$ it is atomless so that $\mathbb{R}(G \cap C)$ is strictly larger than $\mathbb{R}(\emptyset)$; b also forces $\mathbb{R}(G \cap C)$ to have either no predecessors, in which case let x be an arbitrary constructible real number, or else a greatest predecessor, which must be of form $\mathbb{R}(x)$ for some real x (since every member of $\mathcal{F}(B, \chi, G)$ is singly-generated). Now for the Self-Collection axiom to hold with X set to $\mathbb{R}(G \cap C)$, some subset $\mathcal{N}$ of this X's predecessors would have to self-collect into X, but in either case (either no predecessors or greatest predecessor), we would have

$$\bigcup \mathcal{N} \subseteq \mathbb{R}(x) \subset \mathbb{R}(G \cap C),$$

violating clause (iv) of the definition of self-collection.

Lemma 7.1 *No algebra B_x in a RIST-B model has the bounded-predecessors property.*

Let $\{B_x : x \in M^\diamond\}$ be a RIST-B model, and fix an arbitrary point $x \in M^\diamond$. It suffices to find one χ such that the pair (B_x, χ) lacks the bounded-predecessors property. Define $\chi : \mathsf{CSAs}(B_x) \to B_x$ so that $\chi(B_y) \equiv 1$ for all $y \in O_x$ (recall O_x is $M^\diamond$'s intersection with x's open past cone) and $\chi(C) \equiv 0$ for all other $\mathsf{CSAs}(B_x)$-members. Then regardless of G's extension, we have

$$\mathcal{F}(B_x, \chi, G) = \{\mathbb{R}(B_y \cap G) : y \in O_x\}.$$

Because the RIST-B axiom requires for all $y < y'$ that $B_y, B_{y'}$ be guaranteed-forcing-distinct, we have $\mathbb{R}(B_y \cap G) \subset \mathbb{R}(B_{y'} \cap G)$ (with proper inclusion) whenever $y < y'$. Thus a $\mathcal{F}(B_x, \chi, G)$ member always has form $\mathbb{R}(B_y \cap G)$ for some $y \in O_x$, and the set of its predecessors is

$$\{\mathbb{R}(B_{y'} \cap G) : y' \in O_x, y' < y\}.$$

Because O_x is densely ordered, and within O_x we have $y' < y \Rightarrow \mathbb{R}(B_{y'} \cap G) \subset \mathbb{R}(B_y \cap G)$, $\mathbb{R}(B_y \cap G)$ can have no greatest predecessor. $\square$

The flexible-homogeneity problem

B is *homogeneous* if it is isomorphic to each of its principal ideals. B is *ACSA-homogeneous* if it is isomorphic to every principal ideal of every $C \in \mathsf{ACSAs}(B)$. B is *flexibly homogeneous* if:

(1) B is ACSA-homogeneous;

(2) there exists $C \in \mathsf{ACSAs}(B)$ that is guaranteed-forcing-distinct from B;[2]

(3) for each C satisfying (2), $C \subset B$ is not a rigid boolean inclusion;

(4) each isomorphism between any C, C' satisfying (2) extends to an automorphism of B.

If B is flexibly homogeneous then $\mathcal{F}(B, \chi, G)$ cannot satisfy the self-construction axioms; the proof of this given in [6] is long and we will not attempt to summarize it here.

Lemma 7.2 *No algebra B_y in a RIST-B model is flexibly homogeneous.*

Let B_y be an algebra in a given RIST-B model. Choose any $x \in M^\diamond$ such that $x < y$; the RIST-B axiom says explicitly that B_x is an atomless, guaranteed-forcing-distinct complete subalgebra of B_y, and so satisfies clause (2) of flexible homogeneity; furthermore the axiom says that the inclusion is rigid, so that it violates clause (3). □

8 Appendix: Potentially simpler models of RIST

The original paper [4] in which R. Longo introduced simple subfactors may provide a simpler way to transform AQFT models into RIST models. The key idea is that for a certain class of von Neumann subfactor inclusions $\mathcal{R} \subset \mathcal{S}$ that occur frequently in AQFT, there exist canonical subfactors $\mathcal{N}$ that are intermediate (i.e. $\mathcal{R} \subset \mathcal{N} \subset \mathcal{S}$) such that $\mathcal{N}$ is a simple, and therefore rigid, subfactor of $\mathcal{S}$. The key question for our purposes is whether *isotony* is preserved when we pass from the relevant $\mathcal{R}$'s of a particular AQFT model to the corresponding $\mathcal{N}$'s — whether, that is, an inclusion $\mathcal{R}_1 \subset \mathcal{R}_2$ entails $\mathcal{N}_1 \subset \mathcal{N}_2$ for the corresponding rigid subfactors.

Definitions. Let $A \subset B$ be an inclusion of factors. $A \subset B$ is *split* if there exists a type I factor N satisfying $A \subset N \subset B$. If Ω is a vector in the space on which A and B act, we call Ω a *standard vector* for $A \subset B$, and call the triple (A, B, Ω) a *standard* inclusion, if Ω is cyclic for A, B, and $A' \cap B$. (These structures, analyzed in the earlier paper [5], are fundamental to [4].)

Fact 8.1 (see [4], Theorem 4.3) *To any standard split inclusion (A, B, Ω) of infinite factors there corresponds a canonical simple subfactor R of B satisfying $A \subset R \subset B$.*

Note that $A \subseteq R$ due to the way R is built up from a tower $A \subseteq N_1 \subseteq N_2 \subseteq \ldots$ of intermediate factors, as shown in Proposition 4.12 of [4]. □

Definitions.

Let the map $O \mapsto \mathcal{A}(O)$ be an RIST-amenable model [NEED TO DEFINE THIS] of AQFT.

[2]In the notation of [6], "there exists C satisfying $1 \Vdash C \neq_G B$".

$O \mapsto \mathcal{A}(O)$ satisfies the *split-inclusion property* if $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$ is a split inclusion (defined above) whenever O_1, O_2 are open double-cone regions such that O_1's closure (in the usual euclidean topology of M) is contained in O_2.

Fix an arbitrary open double-cone region $M^{outer} \subseteq M$ such that $M^\diamond$'s closure is contained in M^{outer}.

Let $\xi \in H$ be a fixed vector such that for all $p \in M^\diamond$, $(\mathcal{A}(O_p), \mathcal{A}(M^{outer}), \xi)$ is a standard split inclusion of factors.

For $p \in M^\diamond$, let $\mathcal{R}_p$ denote the canonical simple subfactor of $\mathcal{A}(M^{outer})$ provided by Fact 8.1 for the inclusion $(\mathcal{A}(O_p), \mathcal{A}(M^{outer}), \xi)$.

Theorem 8.2 *If the function $\mathcal{A}(O_p) \mapsto \mathcal{R}_p$ with domain $\{\mathcal{A}(O_p) : p \in M^\diamond\}$ is one-to-one and preserves isotony, then $\{\mathcal{R}_p : p \in M^\diamond\}$ is a model of RIST.*

Proof: the only nontrivial thing to show is that if $A \subseteq B \subseteq C$ are von Neumann factors and A, B are both rigidly included in C, then A is rigidly included in B. Would follow if every automorphism of B extended to an automorphism of C. □

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