Bergson's not-even-wrong theory, now with extra math!

Doug McLellan, 2021

We live in a golden age of NEWTs: Not-Even-Wrong Theories of fundamental physics. Of course it is a golden age for real science too, with gorgeous results streaming back daily from the borders of the known. But the methods used to get these results have failed, for nearly a century, to settle the fundamental questions of how quantum theory should be "interpreted" and how it fits with general relativity. One feels there should be a transcendently elegant answer here, an answer that "rings true to the last drop" as Bertie Wooster once put it (in a different context), yet the scientific mainstream seems to offer nothing but kludges. And so a niche has opened up for NEWTs claiming to show how everything will snap perfectly into place. We introduce one such NEWT here: the *Bergsonian axioms project*.

As no one should be expected or encouraged to peruse more than a small fraction of the NEWTs on offer, it behooves the peddler of each new one to begin by identifying the concerns that motivate it — the itches it purports to scratch — lest perusers' valuable time be diverted away from more entertaining or thought-provoking fare. What motivates the Bergsonian axioms is the eponymous philosopher's key insight: *there is no fixed totality of possibilities; the world must be the continual creation of new possibilities.* Henri Bergson was led to this insight by paradoxes that arise when we picture the totality of *biological* or *mental* possibilities as fixed. We wish to deploy the same insight against paradoxes that arise when we picture the universe of *mathematical* possibilities as fixed. This requires a coherent alternative picture of a growing mathematical universe, which is what the Bergsonian axioms supply. We have not yet found a structure that obeys these axioms, but, as we will explain below, a structure that did so would have to resemble models of contemporary physics in so many ways that we are moved to guess that our world *is* such a structure. The NEWT that unpacks this guess may not only vindicate Bergson's original insight, but explain quite directly why the world must evolve through the emergence of random quantum states that determine its curvature.

A one-paragraph outline of the project.

Bertrand Russell sparked a crisis in mathematics by pointing out contradictions involving sets that risk being members of themselves, notably the set-of-all-sets. The consensus "solution" has been to say that all sets taken together form a "class" rather than a "set," but this merely sweeps the contradictions under the rug. Henri Bergson's possibility-creation insight suggests a more robust solution: if new sets emerged over time, the set-of-all-sets *now* could be a member, not of itself, but of a larger set-of-all-sets *later*. This approach only makes sense, though, if we have a way to understand the universe of mathematical possibilities as growing in an organic, non-arbitrary way. The axioms defining a *Bergson history* (stated in full in the Appendix) show the way. The “points” that constitute a Bergson history can be considered snapshots of the mathematical universe’s growth, or more formally as models of the set theory ZF^- (although for technical simplification the axioms identify each point with only the real numbers in such a model — details and references will be given below). A Bergson history is self-constructing in the sense that nothing outside of it guides or feeds its growth: nothing is given in advance, each point is just the closure under definable operations of the collection of all sets that emerged at previous points, and a new point will emerge when and only when “vanishingly little work” would be required for this to happen. We do not yet know whether any structure can satisfy these axioms. To obtain candidate structures we use a recipe (in fact a set-theoretic forcing construction) whose inputs are a boolean algebra B and a function χ that determines which of B’s subalgebras will contribute points to the Bergson history. Alas, none of the most popular boolean algebras works when used as this recipe’s B. An analysis of what goes wrong with them indicates that we need a bespoke B made with “rigidly- and densely-nested subalgebras.” To our knowledge the only potential source for such a thing is algebraic quantum field theory. If an AQFT-derived Bergson history pans out, the new sets that emerge at its points will be random states on algebras of observables. This raises the question whether our own universe, which seems to evolve through the emergence of just such states, could be a Bergson history. We believe it can, because the modifications AQFT would need in order to yield Bergson histories would also explain why (and how) discontinuous measurements and space-time curvature should figure into it. First, to overcome Russell’s paradox, B would need to be modified to incorporate certain “cardinal collapsing” subalgebras, scattered sparsely among its other subalgebras. If it turns out (as we expect) that these collapsing subalgebras must be rigid, then by construing the corresponding points of the Bergson history as measurement events we could obtain a compelling interpretation of quantum measurement as symmetry-breaking. New states would be created continually, but symmetries at *non*-measurement points would distribute the new states that emerge there throughout the state space so that they could only be regarded as new “possible states”; only at a measurement point would the corresponding subalgebra’s rigidity break the symmetry, resulting in a unique new state, the measurement outcome. Second, in order for the axioms to be satisfied, χ will likely need to be nontrivial, meaning that it will pick out a generic proper subset of B’s subalgebras to furnish the Bergson history’s points; since the subalgebras of an AQFT-derived B are ordered as a space-time manifold, we can easily imagine such a χ picking out a generic *sub*-manifold. This recalls a common way of introducing curvature into space-time (e.g. in de Sitter models), namely by embedding our four-dimensional space-time into a higher-dimensional manifold. But unlike de Sitter models, a Bergson history built this way would not have a pre-defined curvature; its curvature would be determined by its random quantum states, hence by the positions and momenta of the particles these states govern (or incarnate), just as Einstein's equation requires in a non-empty universe. Thus the Bergsonian axioms do not just promise a flashy mathematical makeover of physics. Asserting that the mathematical and physical universes are the same thing, they promise to show that it must incorporate randomly arising quantum states that determine its curvature, *on pain of logical contradiction.*

A more leisurely exposition.

There is a popular class of NEWTs that try to sell us some mathematical structure X as the one that really ought to underlie the universe, in light of the impressive contrast between X's own spareness and its outputs' prodigious verisimilitude — what the French would call X's *bon rapport qualité-prix*. The Wolfram Physics Project is probably the purest example of this class of NEWT. The Bergsonian axioms project is (we would like to stress this) not in this class. Its axioms arise from a very different kind of concern, having to do with the foundations of mathematics, specifically with the set-of-all-sets paradoxes discovered by Russell and Cantor. (In Russell's version one considers the property "is a set that doesn't have itself as a member": one asks whether the set of all sets with this property has this property; if it does then by definition it doesn't, and vice-versa.) The consensus solution here is essentially to command, "Thou shalt not say that the collection of all sets is itself a set; thou shalt call it a proper class." This solution works in the sense that it keeps contradictions out of mathematics, but it is obviously just a dodge. Bergson's insight promises a true solution to the paradoxes — for if sets are among the new possibilities that arise over time, the set-of-all-sets *now* could be a member of a larger set-of-all-sets at some *later* point.

The question then is *how* new sets could arise over time (assuming we wish to remain mathematical "realists" and not retreat to the "intuitionist" position that numbers and sets are ultimately figments of our imagination). It is no good to simply assert that sets pop into existence and to refuse further explanation; as a "resolution" of Russell's paradox this would obviously be no improvement over the class/set dodge. What we need is a process in which new sets could be plausibly described as growing in an organic and non-arbitrary way. This need inspires the idea of a *Bergson history*, of which we now give a brief account.

A *model* of a formal theory, i.e. of a given list of axioms, is a structure that obeys those axioms. Those who take the mathematical universe to be timelessly fixed generally consider it to be a model of the set theory ZF^-. (There is more controversy over whether it should also obey the non-constructive "Powerset" and "Choice" axioms, which together with ZF^- constitute the theory ZFC.) We ourselves grant that the mathematical universe should have this form *at every point of its growth process*. Therefore a candidate $\mathcal{F}$ to be a Bergson history will be a family of "points," understood to mean models of ZF^-.[1] And a such a candidate $\mathcal{F}$ will actually succeed in being a Bergson history if it is a model of the following three axioms:

[1] Actually our axioms state that each member of a Bergson history will be a *continuum*, meaning the collection of all real numbers in some ZF^- model. We will ignore this technicality until our discussion of our NEWT's ontology.

• *Self-collection axiom:* A ZF^- model X is a point in $\mathcal{F}$ if and only if it is the closure under definable operations of the union of some smaller points in $\mathcal{F}$ that "self-collect," in the sense that they "naturally form a collection" requiring "vanishingly little work" to undergo this union-and-closure procedure.
• *Foundation axiom:* No sets slip in unaccountably during the course of $\mathcal{F}$'s growth.
• *Anti-Paradox axiom:* There exist points $X, Y \in \mathcal{F}$ such that $X \in Y$.

The formal versions of these axioms are developed in the paper "Self-constructing continua" [13], which is our project's mathematical core. We restate the axioms here in the Appendix — do have a look at them, for it will be helpful to proceed with some confidence that they faithfully capture this intuitive idea of a self-constructing mathematical universe, and that they do not smuggle in anything overtly physics-related.

The recipe for building Bergson histories.

The next question is whether Bergson histories so defined can exist. Set-theoretic *forcing* provides a recipe that outputs candidates to satisfy the axioms; we will now give the recipe along with a high-level explanation of it (the details, again, are in [13]).

The basic idea of forcing is to start with a *ground model* M of the set theory ZFC, which can be thought of as "the totality of all mathematical possibilities," and then to prove it logically consistent that further possibilities, called *generic sets*, could be added to M, yielding a larger totality $M[G]$, the *generic extension.* It is important here that we use the boolean-algebraic formulation, as opposed to the superficially more general "poset" formulation, of forcing. In this formulation one chooses an infinite complete boolean algebra B in the ground model M, to use as a kind of template that will determine the properties of the new sets. (Note that "boolean algebra" denotes the well-known set of logical laws, but when we speak of "a" boolean algebra, we mean a particular model of these laws; the taxonomy of infinite boolean algebras is enormously complicated and is by no means fully mapped out today.) Having chosen B, one postulates a new subset $G \subseteq B$ called a *generic filter* on B.[2] Following our principle that we must begin without anything inexplicable given to us in advance, we will always take our ground model to be the constructible universe L (a.k.a. the "minimal model of set theory"); thus our generic extensions will always have form $L[G]$.

Within such an $L[G]$, every potential "point" we will consider will be, for some set $x \in L[G]$ of natural numbers, "the collection of all sets constructible from x." The best-known formalization of this, namely the "relative constructible hierarchy" $L[x]$, is not quite what we need here. $L[x]$ is defined as the union of all levels $L_\alpha[x]$ of the relative constructible hierarchy, where α ranges over "all the ordinals."[3]

[2] More accurately it is generic on the set of B's nonzero members, but we will ignore this throughout.

[3] The ordinals start with natural numbers $0, 1, 2 \ldots$ and then continue to $\infty, \infty+1, \infty+2, \ldots, 2\infty, 2\infty+1, \ldots, \infty^2, \ldots, \infty^3, \ldots, \infty^\infty, \ldots$. (Although the symbol ω is conventionally used instead of ∞.) An ordinal can be considered "a way of putting things in linear (total) order so that there are no infinite decreasing sequences;" in the context of set theory, which is where we will remain, each ordinal has a canonical set

But the idea that "all the ordinals" are "given in advance" runs counter to our principle that nothing should be so given. We instead define an ordinal α to be "constructibly accessible from x" if it is countable in $L[x]$, and we then formalize "the collection of all sets constructible from x" as $L^{-}[x]$, defined as the union of the levels $L_\alpha[x]$ of the relative constructible hierarchy for which α is constructibly accessible from x.

More generally, if A is any set that is interconstructible with a set x of natural numbers in the sense that $L[A] = L[x]$, we define $L^{-}[A]$ to be $L^{-}[x]$ (of course, to show that this is a legitimate definition requires showing that it does not depend on the choice of x here).

The candidates $\mathcal{F}$ having points of form $L^{-}[x]$ that can be obtained by forcing over L all come from the following recipe:

$$\mathcal{F}(B, \chi, G) \equiv \{L^{-}[G \cap C] : C \text{ is a subalgebra of } B \text{ and } \chi(C) \in G\},$$

where B is a complete boolean algebra, χ is a function from the set of B's subalgebras into B, and G is a generic filter on B. The key things to note here are:

- The members of $\mathcal{F}(B, \chi, G)$ are points $L^{-}[G \cap C]$ corresponding to certain subalgebras C of B.
- χ is what determines (along with G, if χ is nontrivial) which subalgebras of B will contribute points to $\mathcal{F}(B, \chi, G)$.
- The inclusion ordering of points "mostly agrees with" the inclusion ordering on corresponding subalgebras: if $C \subseteq D$ then $L^{-}[G \cap C] \subseteq L^{-}[G \cap D]$, and for purposes of the present overview we can pretend the converse holds too.[4]
- Since these points are meant to be "snapshots of a mathematical universe growing over time," we use temporal vocabulary in discussing the inclusion relation on them, e.g. when $L^{-}[G \cap C] \subset L^{-}[G \cap D]$ and both $\chi(C)$ and $\chi(D)$ are in G, we call $L^{-}[G \cap C]$ a point in the "past" of $L^{-}[G \cap D]$.

The task of constructing a Bergson history thus reduces to choosing a suitable boolean algebra B and function χ such that $\mathcal{F}(B, \chi, G)$ will satisfy the Bergsonian axioms. (Note we cannot "choose" G; its genericity is a kind of randomness that puts it beyond our control.)

When the author first conceived of Bergson histories, he hoped to establish that a wide range of inputs B and χ would work, and then, to the extent that a connection with physics seemed desirable, to pass the whole project off to smarter folks who could integrate it with the relevant theories. If things had worked out accordingly, Bergson histories might have been established as a niche philosophical NEWT, but would have imposed too few constraints on theories to yield any scientific or mathematical insights.

that represents it (due largely, as it happens, to von Neumann). 0 is represented by the empty set $\{\}$, 1 by $\{\{\}\}$, 2 by $\{\{\}, \{\{\}\}\}$; in general the representative set for α is the set of representatives of all smaller ordinals.

[4]The converse does not hold in general; $L^{-}[G \cap C]$ will remain the same if we "add" or "remove" finitely many C-elements, or change C as radically as we like within the part of B disjoint from G.

It turns out, however, that the usual off-the-shelf boolean algebras, namely Cohen algebras and the measure algebras used for random-real forcing, do not work when used as B in the $\mathcal{F}(B,\chi,G)$ recipe. It is shown in [13] that such algebras fail essentially because they have too many symmetries (automorphisms) — and in particular because they have too many automorphisms that leave relevant subalgebras fixed.

The need for boolean subalgebras that are both rigidly and densely nested.

Since the notions of rigidity and of rigid inclusion are at the core of our mathematical challenge, it is worth illustrating them more vividly. An *automorphism* of a structure S is a way of mapping S onto itself ("twisting" it) in such a way that certain specified relations are preserved. A structure is *rigid* if it has no automorphisms except the trivial one that maps everything to itself. A substructure T of S is *rigidly included* in S if no nontrivial automorphism of S leaves T fixed. Imagine for example a metal ring with gear teeth pointing inward, inside which is a much smaller disk T having the same center and outward-pointing teeth; call the whole structure S. If the teeth of the disk and the ring do not touch, so that the ring and the disk can rotate independently, T is not rigidly included in S. But imagine that the axis of the disk T is offset so that its teeth mesh with the ring's. Neither the ring nor the disk is rigid, because both can still be rotated, but the disk T is now rigidly included in the structure S, because the ring cannot rotate without rotating the disk. The notion of rigidly-included boolean algebras is analogous to this geometric picture, but examples are harder to produce.

It is shown in [13] that $\mathcal{F}(B,\chi,G)$ cannot be a Bergson history if B lacks rigidly-included subalgebras and has a few other properties. Loosely speaking, then, a good candidate B to yield a Bergson history ought to have rigidly-included subalgebras. The heavy-handed way to make this happen is to make B rigid itself (for if B has no nontrivial automorphisms at all, then it obviously has no nontrivial automorphisms that fix a given subset $C \subseteq B$). Rigid boolean algebras do exist, as does a rigid algebra B with a chain of rigid subalgebras $C_0 \subset C_1 \subset C_2 \subset \ldots \subset B$. It is shown in [13] that when we apply our recipe to one instance of the latter case, to obtain the set of points

$$L^-[G \cap C_0] \subset L^-[G \cap C_1] \subset L^-[G \cap C_2] \ldots \subset L^-[G],$$

the Bergsonian axioms can hold at the point $L^-[G]$. However, the axioms will fail at every earlier point $L^-[G \cap C_n]$. Every such point (for $n > 0$) has an immediate predecessor point $L^-[G \cap C_{n-1}]$ that includes all the points in $L^-[G \cap C_n]$'s past. Thus the union of all the sets from all of $L^-[G \cap C_n]$'s past points is just $L^-[G \cap C_{n-1}]$, which is already closed under definable operations; so if $L^-[G \cap C_n]$ is to be "just the closure under definable operations of the sets that arose previously," as the Bergsonian axioms demand (in a formalized way), then it would have to equal $L^-[G \cap C_{n-1}]$, which it does not.

The natural way for a candidate Bergson history to get around this problem is for its points to be *densely ordered*, in the sense that whenever $L^-[G \cap C]$ is a point in $L^-[G \cap E]$'s past, there is a point $L^-[G \cap D]$ strictly in between them. Since the temporal ordering of points ("mostly") agrees with the inclusion ordering of their corresponding subalgebras, this would require B to have *densely nested subalgebras.*

The problem we face now is that it is extremely difficult, or maybe impossible, to construct a complete boolean algebra with rigid complete subalgebras that are densely nested in the way we need. (The author has consumed a great deal of paper and coffee trying to combine some ideas from the literature to make this work, without success.)

There is, however, a more subtle strategy we might turn to, which is to construct a boolean algebra with densely-nested subalgebras that are *rigidly included*, but not themselves rigid. We do not know of any examples of this in the boolean-algebra or forcing literature, but through assiduous research (i.e. googling random combinations of terms) we have unearthed a closely analogous structure in the field of operator algebras, namely the *simple subfactors* introduced by R. Longo in the 1980s (see [8]). These are not boolean algebras, but nested C*-algebras $\mathcal{R} \subset \mathcal{S}$ such that no *-automorphism of $\mathcal{S}$ leaves $\mathcal{R}$ fixed. We have isolated in [10] a conjecture under which we can obtain rigidly-included-but-nonrigid boolean algebras from the projection operators in such $\mathcal{R}$ and $\mathcal{S}$.

The examples of simple subfactors that Longo presents in his initial paper [8] are not densely nested. However, he has pointed out to the author in correspondence that densely-nested systems of simple subfactors are likely obtainable from certain models of algebraic quantum field theory, using the technique developed in [9]. Our NEWT is essentially an attempt to follow through on this suggestion; to do so we need some understanding of AQFT.

A précis of AQFT.

The structure from which we hope to wring a Bergson history is a model of algebraic quantum field theory. It is necessary to grasp the basic contours of this theory in order to understand the NEWT we are building atop it. Fortunately, its basic shape is almost as easy to sketch as its inner workings are difficult to untangle. (For more in-depth introductions see [4] or [5].)

Recall the basic structure of quantum mechanics: a system is represented by a complex Hilbert space H; yes/no propositions about the system (like "the electron's velocity in the y-direction is between 5 and 10 cm/s") are identified with projection operators on H, or equivalently with the closed linear subspaces of H which are these operators' ranges; the system's state is (ignoring superpositions) represented by a unit vector $v \in H$; the probability that a proposition represented by projection P will hold true when measured is $\|Pv\|^2$. Under the usual interpretations presented or implied in textbooks, such a measurement makes the system jump discontinuously into the state represented by the unit vector $Pv/\|Pv\|^2$ (or by $(1-P)v/\|(1-P)v\|^2$, if the outcome was "no"). Under the "Heisenberg picture" of system dynamics, the system's state does not change over time except upon measurement events; instead, it is the identification of propositions with projections meant to represent them that changes over time, in accordance with Heisenberg's formula. (The equivalent "Schrödinger picture" reverses this, leaving operators fixed and having the state vector evolve according to Schrödinger's equation.) Measurements in general (i.e. beyond tests of yes/no propositions) are represented by self-adjoint linear operators on H. The algebra $\mathcal{B}(H)$ of all *bounded* linear operators on H is of particular importance.

AQFT can profitably be considered an answer to the question of how this Heisenberg picture might be ported into a special-relativistic context. In such a context, the identification of physical propositions with operators on H ought to depend on space as well as on time; for example, the proposition "the electron's speed relative to our coordinates is between 10 and 12 cm/sec" should be identified with two different projection operators in two spacelike-separated regions of spacetime. AQFT is essentially a covariant way of assigning to each Minkowski-space region the set of operators representing measurements that are possible therein. In particular, the observables in a bounded region Z are represented by a *von Neumann algebra* $\mathcal{R}(Z)$ of operators, meaning a subalgebra of $\mathcal{B}(H)$ that is closed with respect to a certain topology. A key consequence of this closure is that a von Neumann algebra's projection operators always form a bounded complete lattice. Moreover, $\mathcal{R}(Z)$'s projection lattice will typically be *atomless* (have no minimal non-null members), unlike $\mathcal{B}(H)$'s lattice in which every projection with one-dimensional range is minimal. This will be crucial to us, because to be usable in a forcing construction, a lattice must be atomless.[5]

Intuitively, the algebras of observables $\mathcal{R}(Y)$ and $\mathcal{R}(Z)$ in nested regions $Y \subseteq Z$ should be correspondingly nested, i.e. $\mathcal{R}(Y) \subseteq \mathcal{R}(Z)$; this is just to say that when we expand a region we can observe more things in it. We must now say something about *states* on these algebras. In the theory of operator algebras, a state ϕ generalizes the familiar notion of a vector state from quantum theory; ϕ can be thought of an assignment of probabilities to all physical propositions, or more accurately to operators that represent them. If ϕ is a state on $\mathcal{R}(Z)$ and $Y \subset Z$, then ϕ's restriction to $\mathcal{R}(Y)$, written $\phi \restriction \mathcal{R}(Y)$, will be a state thereon. *But ϕ will not in general be determined by its restriction to the subalgebra $\mathcal{R}(Y)$* — many different states γ on $\mathcal{R}(Z)$ may satisfy $\gamma \restriction \mathcal{R}(Y) = \phi \restriction \mathcal{R}(Y)$. And these states may differ sharply in the probabilities they assign to projection operators not in $\mathcal{R}(Y)$, corresponding to outcomes of possible measurements outside of Y. To take an extreme example, which will be important later when we discuss our interpretation of measurement, we may have distinct states γ, ϕ on $\mathcal{R}(Z)$ such that γ and ϕ have the same restriction to $\mathcal{R}(Y)$, which is interpreted there as a low-momentum-uncertainty state for one of the particles being modeled, but also such that γ is a low-*position*-uncertainty state for the same particle within a region lying in the future of Y, while ϕ is still a low-momentum-uncertainty state there.

Obtaining a candidate Bergson history from an AQFT model.

Under a certain Main Conjecture, a complete inclusion of boolean algebras can be obtained from an inclusion of von Neumann algebras. The procedure can be generalized to derive a

[5] One of the most remarkable facts that emerged during the development of AQFT was that the observable algebras of local regions are type III von Neumann algebras, whereas the algebras $\mathcal{B}(H)$ governing standard (nonrelativistic) quantum mechanics are type I. A concise account of this fact and of the upshot of the difference between the two types is given in J. Yngvason's paper [16].

densely-nested system of boolean algebras from the densely-nested system of von Neumann algebras in an AQFT model. The details are worked out in [10]; we will omit them here and simply state the main conclusion. Assume henceforth that the Main Conjecture holds. Then there exist a Minkowski space M, and an AQFT setup that maps the open past cones O_p of points $p \in M$ to the (von Neumann) algebras of observables $\mathcal{R}(O_p)$ in those cones; this yields a boolean algebra B and subalgebras $C_p \subseteq B$ in the following way: B is the boolean completion of $\mathcal{R}(O_z)$'s projection lattice for some arbitrary fixed point $z \in M$, and for all $p \in O_z$, C_p is (the image of a canonical complete embedding into B of) the boolean completion of $\mathcal{R}(O_p)$'s projection lattice.

The upshot for us is that, if we feed this B into our $\mathcal{F}(B, \chi, G)$ recipe, with any generic filter G and a χ that picks out only the subalgebras C_p just defined (for all $p \in O_z$), *there will be an order-isomorphism between the points p in the past cone O_z (with the Minkowski space's usual temporal ordering) and the points $L^-[G \cap C_p]$ in our Bergson history candidate (ordered by inclusion).*

We now have a system of densely-nested boolean subalgebras; unfortunately, an additional layer of complexity is needed to ensure that they are also *rigidly* nested. If we begin with a standard AQFT model the resulting inclusions $C_p \subset C_q$ will not be rigid, because the von Neumann algebra inclusions $\mathcal{R}(O_p) \subset \mathcal{R}(O_q)$ are not *simple* inclusions in the sense of [8]. A more recent paper [9] by R. Longo and collaborators provides a tool for expanding the algebras $\mathcal{R}(O_p)$ so that the inclusions will be simple; see [10] for details on how to apply the tool.

Retroactively assuming, then, that our original AQFT model was built with this tool, we have plausible candidates for B and χ. Now the remarkable thing about a generic filter G on this B is the following.

Theorem A. *A generic filter G on the projection lattice of a separably-acting von Neumann algebra $\mathcal{R}$ having no minimal projections (or on the boolean completion of this lattice) is canonically interdefinable with a generic normal state ϕ_G on $\mathcal{R}$.*

This is the main result of [12], which also explains why such states ϕ_G can be regarded as generalizations of random real numbers. Let ϕ_G be the generic state canonically obtained from the filter G on our B; for each point p in the past cone O_z, $G \cap C_p$ will be canonically interdefinable with ϕ_G's restriction to p's past-cone algebra $\mathcal{R}(O_p)$. So Bergson history candidates derived from AQFT along these lines — and we have no other plausible candidates — *are mathematical universes that grow through the emergence of random states on algebras of quantum observables.* Since our physical universe manifestly grows through the emergence of such states, this is what suggests the leap from the purely abstract set-theoretical cerebrations that we have been engaged in so far, to a theory of physics. We now consider the most basic question about what this leap would entail.

The ontology of the Bergsonian axioms NEWT.

A primary task, arguably the primary task, of a Not-Even-Wrong Theory is to say *what there is*: monads, atoms, waves, or what. A great divide in NEWT taxonomy separates those that deal in configurations of *stuff* from those that, in the Pythagorean "all is number" tradition, renounce stuff and weave the world out of pure abstract configurations, or "information." Our Bergsonian NEWT falls on the latter side. Of course this side has subdivisions too: we may ask *which* pure configurations make up reality and expect different answers from different NEWTs. (A string theory, for instance, might attribute reality to "branes" in a continuous manifold, while another NEWT might admit only a discrete, locally-finite space-time.) It is with respect to this question that our NEWT distinguishes itself most radically: we escape it entirely, using Bergson's signature move. We deny that there is one timelessly fixed totality of possible configurations, some of which are for some reason anointed real. Instead we understand the world as the continual creation of genuinely new abstract possibilities, and understand "being *real* at this moment" to mean "being *newly possible* at this moment." And if some structure (say a real number x) is newly possible, then everything interdefinable or interconstructible with it (like the real number $47x + 2$, or (say) a normal state on a von Neumann algebra encoded by x's decimal expansion) is newly possible too, and we recognize no ontological priority of one such structure over another. For us, what characterizes a point in space-time is not any *particular* structure living within a model of set theory; it is the whole model of set theory that constitutes what is possible there. In this way we collapse the notion of reality into a dynamic conception of possibility, formalizing Bergson's position in "The possible and the real" [2], which can be recommended as the canonical exposition of this way of thinking.

Although we give no *ontological* priority to any set within a Bergson history point $L^-[G \cap C]$, there are *practical* reasons to focus on some sets in $L^-[G \cap C]$ that are interdefinable with it, and can therefore be used as "stand-ins" for it. For example, since we have stipulated that all our $G \cap C$ will satisfy $L[G \cap C] = L[x]$ for some set x of natural numbers, we could use "the set of all sets x of natural numbers such that $L[x] = L[G \cap C]$" as such a stand-in. Or, by construing sets of natural numbers as real numbers (as is common in set theory), we could use "the set of all real numbers in $L^-[G \cap C]$," i.e. its *continuum.* In fact, our statement of the Bergsonian axioms (see the Appendix) uses continua as the members of a Bergson history, and considers the set-theory models they construct as secondary.

Phrasing the Bergsonian axioms in terms of their points' continua makes them more immediately comprehensible. But because we have confined our attention to points obtained by forcing, so that they have form $L^-[G \cap C]$, we might well ask why $G \cap C$ couldn't be used as a more natural stand-in set. The answer is that, although $L^-[G \cap C]$ is obviously definable (indeed defined) from $G \cap C$, the converse does not hold in general. In fact, our boolean algebra B, which we have defined from a von Neumann algebra's projection lattice, is like the standard Cohen and random-real algebras in that a generic filter $G \subset B$ will never be definable (without parameters) in $L^-[G]$.

We can, however, use the following as a natural interdefinable stand-in for $L^-[G \cap C]$: *The set of all generic filters F on boolean subalgebras of B such that $L^-[F] = L^-[G \cap C]$.*

This is an equivalence class of filters that construct the same set-theory model, and it is indeed the most useful stand-in to use when reasoning about Bergson histories obtained through forcing. We will call this the *Bergson state class* of $L^{-}[G \cap C]$, and call its members *Bergson states*. The motivation for this "state" terminology is Theorem A above, which entails that (when our AQFT-derived B is used) a Bergson state will induce a state (in the usual AQFT sense) on an associated algebra of observables.

Cardinal-collapsing boolean algebras and the interpretation of measurement.

Returning to our main task of building a Bergson history $\mathcal{F}(B, \chi, G)$, we must now admit that two issues make the AQFT-derived B and χ we defined above unlikely to work in their current form. This is not disheartening, though — quite the opposite, since these issues point to fixes that would make for a far more compelling NEWT. The modification that B apparently needs to overcome the first issue would naturally explain quantum measurement; and the modification that χ apparently needs to overcome the second issue would explain how curvature in the Bergson history is determined by random quantum states. We will try to convey how compelling these explanations are — or would be, if the math worked out — as motivation for investigating whether the math actually does work out. We begin with the first issue.

Recall that the original purpose of Bergson histories was to provide a robust solution to the Russell-type paradoxes, by allowing the totality of possible sets at one point to form a set belonging to the totality at a later point. (This is formalized in the third Bergsonian axiom.) Given our stipulation about the types of points we are considering, this means $L^{-}[G \cap C] \in L^{-}[G \cap D]$ for some subalgebras C, D of B. Next recall how $L^{-}[G \cap C]$ was defined, in terms of the ordinals "constructibly accessible from $G \cap C$," meaning *countable* in $L[G \cap C]$. It follows from this definition that if $L^{-}[G \cap C] \in L^{-}[G \cap D]$, then the set α of all ordinals in $L^{-}[G \cap C]$ — which is the first uncountable ordinal in $L[G \cap C]$ — is a member of $L^{-}[G \cap D]$, and is therefore countable in $L[G \cap D]$. In the parlance of forcing, $G \cap D$ has "collapsed" α, the first uncountable ordinal in $L[G \cap C]$. (More explicitly: it is impossible to construct from $G \cap C$ any one-to-one correspondence between the natural numbers and the members of the set that canonically represents α, but such a correspondence is constructible from $G \cap D$.)

Now consider the second Bergsonian axiom, which states informally that no set slips in "inexplicably" in the course of a Bergson history's growth. More formally, it states that whenever a set x belongs to some point Z, it belongs to some *minimal* point $Y \subseteq Z$ (i.e. no point strictly in Y's past has x as a member). Applied to cases where x is an ordinal, this suggests the following terminology: a *paradox-escape point* in a Bergson history is a point having an ordinal member that is not a member of any smaller (earlier) point in the history.

We noted above that the points of a Bergson history must be densely ordered (or, at any rate, that dense ordering is the obvious way around the "immediate predecessor" problem that would thwart satisfaction of the axioms). This has the following important consequence for paradox-escape points:

Theorem B. *Assuming a Bergson history's points are densely ordered, not all of them are paradox-escape points; moreover, paradox-escape points cannot be order-dense in any interval of the history.*

The proof here is short enough to give in full. Assume a history with densely-ordered points, i.e. for any distinct $X \subset Y$ in the history there is a third point Z such that $X \subset Z \subset Y$. If paradox-escape points were order-dense in some interval, there would exist an infinite sequence $P_0 \supset P_1 \supset P_2 \supset \dots$ of paradox-escape points therein. By definition of paradox-escape point let α_n denote the smallest ordinal that is in $L^{-}[P_n]$ but not in $L^{-}[P_{n+1}]$. Then the α_n would form an infinite descending sequence of ordinals below α_0, violating the definition of ordinals.

Thus, informally speaking, the paradox-escape points must be "scattered sparsely" among the "regular" points. This is already suggestive of measurement events scattered among the "regular" intervals of a quantum system's lifetime, during which it evolves continuously. But we are just coming to the main motivation for identifying paradox-escape points with measurement events.

Suppose that $L^{-}[G \cap C]$ is a paradox-escape point, and that $\alpha \in L^{-}[G \cap C]$ is the least ordinal that is not a member of any earlier point in the Bergson history. Then C is a boolean algebra that can make α countable, but for no strictly smaller point $L^{-}[G \cap D] \subset L^{-}[G \cap C]$ does $G \cap D$ make α countable. The most direct way to arrange this is for C to be a *minimal α-collapsing algebra*, meaning that regardless of a generic filter G's extension, α will be countable in $L[G \cap C]$ and uncountable in $L[G \cap D]$ for any proper complete subalgebra D of C.

Now collapse algebras come in many varieties. The most commonly used one (sometimes called "Lévy collapse") is a close cousin of the Cohen algebra; and like the Cohen algebra, it abounds in symmetries (automorphisms). We therefore hypothesize confidently that it will fail in our $\mathcal{F}(B, \chi, G)$ recipe. But we will offer the following stronger hypothesis too, with correspondingly weaker confidence: *the only minimal collapse algebras that will work as subalgebras of B are rigid.*[6] The intuitive idea here is that a collapse algebra C is in a sense "very large" compared to the non-collapse subalgebras $C' \subset C$ that it properly includes, so that if it were possible to "twist" C with *any* nontrivial automorphism, C would have enough "slack" to be twisted *without twisting C' at all*, so that C' could not be rigidly included in C. This is of course highly informal, and, in contradistinction to most of the other informal mathematical claims we have made above, it has not been investigated in any rigorous way. But let us see where this stronger hypothesis, if true, would lead.

Suppose we modify our AQFT-derived B so that it has rigid minimal collapse subalgebras that themselves have "regular" subalgebras derived as above from projection lattices of

[6] It is worth pointing out that rigid minimal collapse algebras do exist: see [1], [7]. For results on embedding arbitrary boolean algebras into rigid algebras, see [15], and for embedding into rigid collapse algebras, see [14].

local observable algebras. By "zooming in" to one of these rigid minimal collapse algebras and restricting our attention to it, we can assume that B itself is of this type. We can then invoke our recipe to obtain a candidate Bergson history $\mathcal{F}(B, \chi, G)$ with the assumption that $\chi(B) \in G$, so that $L^-[G]$ is a point (the greatest, or "latest" point) in the history. Consider the Bergson state class that is equivalent to (interconstructible with) the point $L^-[G]$. This was defined above as the set of generic filters F on subalgebras of B such that $L^-[F] = L^-[G]$. Obviously G itself is in this class. Since B is a minimal collapse algebra, it necessarily collapses some α that no proper subalgebra C can collapse, so for no generic filter F on any such C can we have $L^-[F] = L^-[G]$. Furthermore, B's rigidity ensures that no generic filter F on B, besides G itself, satisfies $L^-[F] = L^-[G]$. Therefore we have:

Theorem C. If B is a rigid minimal collapse algebra then the Bergson state class corresponding to $L^-[G]$ has only one member, G.

Contrast this to the situation at an earlier "regular" point $L^-[G \cap C_p]$, where C_p is not rigid (being the boolean completion of the projection lattice of the von Neumann algebra $\mathcal{R}(O_p)$). Here the symmetries of C_p will result in the Bergson state class of $L^-[G \cap C_p]$ having infinitely many Bergson states $F \subseteq C_p$, which induce (by Theorem A above) infinitely many different states ϕ_F on the algebra $\mathcal{R}(O_p)$. These can be interpreted as "possible states" on this algebra of observables. By Theorem C, however, there is *only one* Bergson state at the point $L^-[G]$; when we reach this point, we consider all the possible states on $\mathcal{R}(O_p)$ to have been "winnowed out" at $L^-[G]$ except for $\phi_G \restriction \mathcal{R}(O_p)$. Other Bergson states F may have been equally plausible candidates to be the generic filter that "really" generated the Bergson history up to the point $L^-[G \cap C_p]$, but it turned out that none of these others could extend to a Bergson state on all of B that would be compatible with the way the system kept evolving.

Thus our interpretation of measurement at a point amounts to a breaking of symmetry that winnows out possible explanations for how the mathematical universe could have evolved up to past points. This interpretation is available to us because our NEWT characterizes a point not (in general) by one particular state on the AQFT algebra of its past cone, but by an *equivalence class* of such states (those canonically induced by members of the point's Bergson state class). This may seem incompatible with the actual practice of quantum physics, which assigns particular quantum states to a system. But we recall here the main difference between states in AQFT and states in basic non-relativistic quantum theory: an AQFT state assigns probabilities to measurements that may take place in regions throughout space-time; and different states may assign the same probabilities to a past region, while differing on later regions.

What happens *after* a measurement point $L^-[G]$, under our interpretation? Later points will be set-theory models $L^-[G']$ constructed by larger generic filters $G' \supset G$ on larger boolean algebras $B' \supset B$. These larger boolean algebras B' may all be, at least for a while, extensions of B by our "regular" kind of AQFT-derived algebra, with the extension carried out in the well-known sense of iterated forcing. In this case these B' will not be rigid, and once again the Bergson state classes associated with the points $L^-[G']$ will have infinitely many members F, corresponding to "possible states" ϕ_F. *But the rigidity of the subalgebra $B \subset B'$ will presumably ensure that the restrictions of these F to B will always be G.* In other words, the state at $L^-[G]$, having been "measured" there, is fixed forever. This is the importance of the fact we mentioned earlier: on the AQFT algebra of a large region, many different states can have the same restriction to the subalgebra corresponding to a smaller region.

How the need for a nontrivial χ can explain space-time curvature.

We have stressed that for $\mathcal{F}(B, \chi, G)$ to satisfy the Bergsonian axioms, symmetries — more exactly, certain automorphisms — must be eliminated. We have stressed further that when $D \subset C$ are two subalgebras of B contributing distinct points $L^-[G \cap D] \subset L^-[G \cap C]$ to our candidate, C should have no boolean automorphisms (isomorphisms onto itself) that leave D fixed. Unfortunately there is another type of symmetry that threatens satisfaction of the Bergsonian axioms, namely isomorphisms θ that map C onto *another* subalgebra $C' \supset D$ but leave D fixed. (These are instances of B's "partial automorphisms" whose suppression is a main concern of [13].) To see how such an isomorphism θ might arise in our AQFT-derived B, consider points $d < c < c'$ in the Minkowski space M, with past cones $O_d \subset O_c \subset O_{c'}$ whose observable algebras correspond to subalgebras $C_d \subset C_c \subset C_{c'}$ of B; if σ is a conformal "stretching" of O_c onto $O_{c'}$ that leaves O_d fixed, then σ may induce a boolean isomorphism θ of C_c onto $C_{c'}$ that leaves C_d fixed.

However, such isomorphisms θ are only dangerous when they preserve not only boolean structure but also χ, in the following sense: for all subalgebras $E \subseteq C$ such that $\chi(E) \in C$, we have $\chi(\theta[E]) = \theta(\chi(E))$. This fact doesn't help us if we keep our original AQFT-derived B and χ, because our χ took value 1 on all B-subalgebras of form C_a; if θ is the boolean isomorphism induced by the conformal mapping σ suggested above, we will have $\chi(\theta[C_a]) = \theta(\chi(C_a)) = 1$, for all points a having corresponding C_a defined.

To prevent this kind of "boolean+χ isomorphism" we can make χ nontrivial, i.e. make it take values other than 0 and 1. In particular, we can make it take values other than 1 on the subalgebras C_a corresponding to past cones with vertices a. This would make it a generic matter — i.e. a "random" matter that, like the generic states ϕ_G, depends on G — whether a point a in the Minkowski space would have a corresponding point in the Bergson history candidate $\mathcal{F}(B, \chi, G)$ or not. If we use such a χ, then some points in Minkowski space will have corresponding points in $\mathcal{F}(B, \chi, G)$, and some won't, and what makes the selection is the same "random" thing that determines our Bergson states and their corresponding "physical states," namely the generic filter $G \subset B$.

The natural way to set up a χ like this would be to have it pick out a generic submanifold of the Minkowski space M used in our original AQFT setup. We have not attempted to define such a χ explicitly, and do not know how straightforward it would be. But we wish to underline how remarkable it would be if we could deduce not only the possibility, but the necessity, of giving our Bergson history the structure of a submanifold of Minkowski space. For this is a popular way of defining curvature in cosmological models, notably in de Sitter models, and in our case the curvature would depend on the quantum states that arise in our Bergson history. This would provide a lovely answer to the puzzle of how quantum theory and general relativity interact — if the mathematical details worked out.

Conclusion.

True NEWT connoisseurs always keep an eye out for signs of *L'Russe Besuhof syndrome*. Recall how Pierre Bezukhov in *War and Peace* learned through numerology that his own fate would be world-historically intertwined with Napoleon's: each of their names, when scored by a Masonic formula, yielded the biblical Number of the Beast, 666. To Pierre's mind this could not be mere coincidence — and indeed it wasn't, because to make the math work, he had first to give himself the French appellation "Le russe Besuhof" and then to elide a letter from the definite article. A NEWT inventor will sometimes proceed similarly. He (and typically it is a he) insists that his first principles are pristinely *a priori*, based in considerations philosophical, mathematical, or otherwise rarified, and when his principles are revealed to have consequences that agree with contemporary science, he is forced to conclude — against his will, really, given his well-known modesty — that it cannot be coincidence, that his first principles must underlie the physical world after all. But an omniscient narrator could reveal that the NEWT inventor fudged his principles or initial conditions in a way that he half-consciously knew would result, downstream, in better agreement with known science.

Have we leapt too quickly to AQFT in our search for a boolean algebra B that works in our recipe? Are we ignoring other algebras equally suited or better suited to yield Bergson histories? Is ignoring these algebras our way of ignoring the "e" in "le"? We respectfully deny it. The main results about Bergson histories had been derived long before we ever heard of AQFT, which, as we said above, we encountered only when we had run out of other ideas for satisfying the Bergsonian axioms.

No, the real risk to the Bergsonian axioms project is simply that the math will not work out. We mentioned above a Main Conjecture on which our work largely rests. The quickest way to promote our not-even-wrong theory into a wrong theory would be to disprove this conjecture, which is detailed in [10]. It is simple enough to state here:

Conjecture. If $\mathcal{R} \subseteq \mathcal{S}$ is a simple subfactor inclusion (as defined in [8]) of injective von Neumann factors, then every partition of $\mathcal{R}$'s projection lattice is also a partition of $\mathcal{S}$'s projection lattice.

One of the author's main goals in providing this overview of the Bergsonian axioms project is to attract comrades in the effort to confirm or disconfirm this conjecture; he himself is by no means a world-class mathematician, and his inability to decide the issue so far should not be taken as evidence that it is intractable.

The mathematical details of our project can be found in the references listed below, notably "Self-constructing continua" [13]; these have been collected online at bergsonian.org. We would also recommend, for a deeper look at the philosophical aspect of the project, the paper "Bergson's paradox, and Cantor's" [11], and (especially) Bergson's own "The possible and the real" [2]. We have steered away here from this aspect, but we will close by affirming that our project's greatest benefit, should it pan out, would not be a reconciliation of quantum mechanics and general relativity, nor a better understanding of the connection between mathematics and physics. It would be a way to see ourselves as active participants in the world's continual self-creation, rather than in the way that all philosophies positing a fixed totality of possibilities ultimately see us, as passive spectators shunted randomly into various pre-existing branches of the future.

Appendix: The Bergsonian axioms

The idea of a self-constructing family of continua is rooted in the theory of *constructibility*, for which see the first two chapters [3], or Chapter 13 of [6]. In particular it is based on the *relative constructible hierarchy* $L(X)$ seeded with an arbitrary set X (see [6], Definition 13.24). Informally $L(X)$ is the collection of those sets that must exist given that X does (and given all the ordinals, and given that the ZF axioms hold). Formally $L(X)$ is the union of the levels $L_\alpha(X)$, where α ranges over the ordinals, and each level is defined in terms of lower levels exactly as in Gödel's original definition of L, except that the base level $L_0(X)$, rather than being defined as the empty set, is now defined as the transitive closure of $\{X\}$.

The *constructive closure* $\mathbb{R}(X)$ of an arbitrary set X of real numbers is, informally, the set of all real numbers that must exist given that X does; formally, we define $\mathbb{R}(X)$ for any set X as the set of all real numbers in $L(X)$.

A *continuum* is a set X of reals that is constructively closed, meaning $\mathbb{R}(X) = X$.

If moreover $X = \mathbb{R}(x)$ for some real number x, we call X *singly generated*.

A set $\mathcal{N}$ of continua *self-collects into* another continuum X if the following hold:

(i) $X \notin \mathcal{N}$; (X is a new continuum)
(ii) $X, Y \in \mathcal{N} \Rightarrow (\exists Z \in \mathcal{N})(X, Y \subseteq Z)$; ($\mathcal{N}$ is *directed*)
(iii) $\mathcal{N} \in L(X)$; (nothing beyond X is used to construct X)
(iv) $(\neg\exists x \in X)(\bigcup\mathcal{N} \subseteq \mathbb{R}(x) \subset X)$; ($\mathcal{N}$ is unbounded below X)
(v) $X = \mathbb{R}(\bigcup\mathcal{N})$. ($X$ is the constructive closure of $\mathcal{N}$'s union)

A *self-constructing family* $\mathcal{F}$ of continua is one that satisfies the following two axioms:

Self-Collection: $(\forall X \in L(\mathcal{F}))(X \in \mathcal{F} \iff$ some $\mathcal{N} \subseteq \mathcal{F}$ self-collects into X, and $X = \mathbb{R}(x)$ for some real $x)$.

Foundation: $(\forall X \in \mathcal{F})(\forall x \in X)(\exists Y \in \mathcal{F})$ $(x \in Y \subseteq X$, and $(\forall Z \in \mathcal{F})(Z \subset Y \Rightarrow x \notin Z))$.

To state the third axiom requires broadening the scope of our attention from the continua $X \in \mathcal{F}$ to the models of set theory that they generate, each of which can be considered a snapshot at one point in the growth of a self-constructing set universe.

Letting X be a singly generated continuum, as witnessed by $X = \mathbb{R}(x)$ for some real x, and letting $\lambda(X)$ denote the smallest uncountable ordinal in $L(X)$, we define:

$$L^{-}(X) \equiv L_{\lambda(X)}(x).$$

An argument is needed to show that this definition does not depend on the choice of x, but we will omit it here. Note that all sets in $L^{-}(X)$ are countable and that the continuum X is a subset, but not a member, of $L^{-}(X)$. Thus $L^{-}(X)$ violates the powerset axiom and is not a model of ZF. It does, however, satisfy all the other axioms of ZF, which are collectively called ZF^{-}.

A *Bergson history* is a family $\mathcal{F}$ of singly-generated continua that satisfies the Self-Collection Axiom, the general version of the Foundation Axiom below, and the following "Anti-Paradox Axiom."

Foundation Ax. (general version): $(\forall X \in \mathcal{F})(\forall s \in L^{-}(X))(\exists Y \in \mathcal{F})$
$(Y \subseteq X$ and $s \in L^{-}(Y)$ and $(\forall Z \in \mathcal{F})(Z \subset Y \Rightarrow s \notin L^{-}(Z)))$.

Anti-Paradox Ax.: $(\exists X, Y \in \mathcal{F})(L^{-}(X) \in L^{-}(Y))$.

References

[1] U. Abraham, "Minimal Model of '$\aleph_1^L$ Is Countable' and Definable Reals," Advances in Mathematics 55. 75-89 (1985).

[2] H. Bergson, "Le possible et le réel" (1930), published in *La pensée et le mouvant*, 1938.

[3] K. Devlin, *Constructibility*, Springer-Verlag, 1984 (also on projecteuclid.org, as of this writing).

[4] C. Fewster and K. Rejzner, "Algebraic quantum field theory — an introduction," arXiv, 2019.

[5] H. Halvorson, "Algebraic quantum field theory," arXiv arXiv:math-ph/0602036v1.

[6] T. Jech, *Set Theory*, 3rd Edition, Springer, 2003.

[7] V. Kanovei and V. Lyubetsky, "Definable minimal collapse functions at arbitrary projective levels." *The Journal of Symbolic Logic*, 84(1), 266-289.

[8] R. Longo, "Simple injective subfactors." Adv. in Math., 63(2):152–171, 1987.

[9] R. Longo, Y. Tanimoto, and Y. Ueda (2017). "Free products in AQFT." Annales de l'Institut Fourier. 69. 10.5802/aif.3269.

[10] D. McLellan, "AQFT as a possible source of self-constructing continua," 2021.

[11] _____, "Bergson's paradox, and Cantor's," 2020.

[12] _____, "Noncommutative analogs of random real numbers," 2019.

[13] _____, "Self-constructing continua," 2018.

[14] P. Štěpánek, "Cardinal collapsing and ordinal definability," The Journal of Symbolic Logic Vol. 43, No. 4 (Dec., 1978), pp. 635-642.

[15] P. Štěpánek and B. Balcar, "Embedding Theorems for Boolean Algebras and Consistency Results on Ordinal Definable Sets," The Journal of Symbolic Logic Vol. 42, No. 1 (Mar., 1977), pp. 64-76.

[16] J. Yngvason, "The Role of Type III Factors in Quantum Field Theory," arXiv (2004).