Rigid boolean inclusions

Doug McLellan, 2020

Abstract

Boolean algebras $A \subseteq B$ constitute a *rigid boolean inclusion* when there is no nontrivial boolean automorphism of B that leaves each A element fixed. We ask here whether a rigid boolean inclusion $A \subseteq B$ is possible, such that A and B are complete, atomless, nonrigid boolean algebras, and A is a proper complete subalgebra of B. Rigid *-algebraic inclusions $\mathcal{R} \subseteq \mathcal{S}$ of von Neumann algebras were obtained by R. Longo in the 1980s in [3]; we state a conjecture under which a rigid boolean inclusion of the kind just stated could be derived from the projection lattices of $\mathcal{R}$ and $\mathcal{S}$.

1 Introduction

A set B is *rigid* with respect to certain relations defined on it if it has no nontrivial automorphism (preserving those relations). An inclusion $A \subseteq B$ is a *rigid inclusion* if B has no nontrivial automorphism that leaves every A member fixed.

We assume familiarity with the axioms defining a boolean algebra $(B, \wedge, \vee, \neg, 0, 1)$, and with the subset of these axioms defining a bounded lattice $(B, \wedge, \vee, 0, 1)$. In both cases one defines the partial ordering on B by $x \leq y \iff x \wedge y = x$. As we will never have cause to consider multiple lattice structures or boolean structures on the same underlying set B, we will freely conflate B with $(B, \wedge, \vee, 0, 1)$ or $(B, \wedge, \vee, \neg, 0, 1)$.

A bounded lattice B is *complete* when every $X \subseteq B$ has a well-defined join (least upper bound) $\bigvee X \in B$ and meet (greatest lower bound) $\bigwedge X \in B$. An *atom* is a minimal nonzero element. When B is a bounded lattice (or boolean algebra) and $A \subseteq B$, we call A a *bounded sublattice* (resp. *subalgebra)* if A is a bounded lattice (resp. boolean algebra) with the same 0 and 1 as B, under the relations inherited from B. If furthermore $\bigvee X$ and $\bigwedge X$ exist for all $X \subseteq A$ and are the same whether evaluated in A or B, then $A \subseteq B$ is a *complete inclusion*, and A is a *complete bounded sublattice* (resp. *complete subalgebra*) of B.

Main Question: Can there exist a rigid complete boolean inclusion $A \subseteq B$ such that A and B are distinct complete, nonrigid, atomless boolean algebras?

Note that we specify "atomless" here because examples can be constructed easily if A is allowed to have atoms. For example, let C be any rigid complete boolean algebra; let B be the "lottery sum" of two copies of C, having an element $x \in B$ such that $\{y \in B : y \leq x\}$

and $\{y \in B : y \leq \neg x\}$ are both isomorphic to C; let $A \subseteq B$ be the complete subalgebra $\{1, x, \neg x, 0\}$: then A and B are both nonrigid (each having a nontrivial automorphism that exchanges x and $\neg x$), but B rigidly includes A, since the restriction to $\{y \in B : y \leq x\}$ of any nontrivial automorphism of B that fixed x and $\neg x$ would be a second isomorphism between the two copies of C, which is impossible since C is rigid.

This paper conjectures that the Main Question has an affirmative answer, and that an example can be derived from a rigid *-algebraic inclusion $\mathcal{R} \subseteq \mathcal{S}$ of von Neumann algebras, of the kind introduced by R. Longo in [3]. The proposed derivation goes as follows: Let $\mathbb{P}_{\mathcal{R}}$ and $\mathbb{P}_{\mathcal{S}}$ be the projection lattices of $\mathcal{R}$ and $\mathcal{S}$ respectively; let $B_{\mathcal{R}}$ and $B_{\mathcal{S}}$ be their respective boolean completions (equivalently, their regular open algebras under the order topology); and let $B_{\mathcal{R} \subseteq \mathcal{S}}$ denote $B_{\mathcal{R}}$'s image under the natural candidate to be a boolean embedding $B_{\mathcal{R}} \to B_{\mathcal{S}}$. We will isolate a simple conjecture about $\mathcal{R}$ and $\mathcal{S}$ that holds if and only if $B_{\mathcal{R} \subseteq \mathcal{S}} \subseteq B_{\mathcal{S}}$ is a rigid boolean inclusion as demanded by the Main Question.

2 Simple subfactors

The starting point for the derivation just sketched is the theory of von Neumann algebras, with which we assume familiarity. We will cite [2] for basic results of this theory. We assume that all von Neumann algebras considered act on a complex Hilbert space H of countably infinite dimension; they thus have separable preduals.

We recall some terminology for properties of a von Neumann algebra $\mathcal{R}$. $\mathcal{R}$ is a *factor* if its center (the set of all $T \in R$ such that T commutes with every operator in $\mathcal{R}$) is trivial (consists of multiples of the identity operator). $\mathcal{R}$ is *infinite* if there exists a partial isometry $V \in \mathcal{R}$ such that $V^*V = P$ and $VV^* = 1$.

The rigid *-algebra inclusions that we will use are the *simple subfactors* introduced by R. Longo in [3]. They are based in the Tomita-Takesaki modular theory, for a full treatment of which see Chapter 9 of [2]. Let $\mathcal{S}$ be a von Neumann factor acting on Hilbert space H, with separating and cyclic vector $\xi \in H$; let J be the modular conjugation of $\mathcal{S}$ with respect to ξ; let $\mathcal{R}$ be a subfactor of $\mathcal{S}$; then $\mathcal{R}$ is called a *simple subfactor* of $\mathcal{S}$ if the von Neumann algebra $(\mathcal{R} \cup J\mathcal{R}J)''$ generated by the union of $\mathcal{R}$ and $J\mathcal{R}J$ is the whole algebra $\mathcal{B}(H)$ of bounded operators on H. Note that the choice of ξ relative to which J is defined does not affect whether this condition holds.

The main result of [3] is the following.

Simple Subfactor Theorem ([3], Prop. 4.2 and Thm. 4.3) Any infinite factor $\mathcal{S}$ with separable predual includes a simple injective (proper) subfactor $\mathcal{R} \subseteq \mathcal{S}$; every simple subfactor inclusion is a rigid *-algebraic inclusion. □

In order to ensure that the boolean algebra $B_{\mathcal{R}}$ we plan to derive from the subfactor $\mathcal{R}$ in the foregoing theorem is atomless (has no minimal nonzero elements), we need to confirm that $\mathcal{R}$ is not type I (has no minimal non-null projections). The following lemma accomplishes this:

Lemma 2.1. *If $\mathcal{R} \subseteq \mathcal{S}$ is a rigid *-algebraic inclusion of factors with $\mathcal{S}$ not type I, then $\mathcal{R}$ is not type I either.*

We want to show that if $\mathcal{R}$ were type I, there would exist a nontrivial unitary $U \in \mathcal{S} \cap \mathcal{R}'$; the nontrivial automorphism $T \mapsto UTU^*$ induced on $\mathcal{S}$ would then act trivially on $\mathcal{R}$ (since U commutes with $\mathcal{R}$), contradicting the rigidity of the inclusion.

Suppose then that $\mathcal{R}$ is type I. By Theorem 6.6.1 of [2] there exists a *-isomorphism $\phi : \mathcal{R} \to \mathcal{B}(K)$ whose range is the algebra $\mathcal{B}(K)$ of all bounded operators on some Hilbert space K. Let $\{x_i : i \geq 0\}$ be an orthonormal basis for K; for all $i, j \geq 0$, let $W_{i,j}$ be the unique partial isometry in $\mathcal{B}(K)$ that maps x_i to x_j and annihilates everything orthogonal to x_i. Every $T \in \mathcal{B}(K)$ is then a (possibly infinite) linear sum of $W_{i,j}$'s, whose coefficients can be considered entries in T's matrix representation in our basis.

It follows by isomorphism that every $T \in \mathcal{R}$ is a (possibly infinite) linear sum of partial isometries of form $\phi^{-1}(W_{i,j})$.

For $n \geq 0$, let P_n denote the projection $\phi^{-1}(W_{n,n})$.

It is well known (see [2], Lemma 6.5.6) that in a non-type-I factor $\mathcal{S}$, any non-null projection can be "halved," which means, for our P_0 in particular, that there exist in $\mathcal{S}$ a projection $Q_0 < P_0$ and a partial isometry V_0 such that $V_0^* V_0 = Q_0$ and $V_0 V_0^* = P_0 - Q_0$.

Now for $n > 0$ define

$$Q_n \equiv \phi^{-1}(W_{0,n})\, Q_0\, \phi^{-1}(W_{n,0})$$

(note Q_n is a proper sub-projection of P_n), and define

$$V_n \equiv \phi^{-1}(W_{0,n})\, V_0\, \phi^{-1}(W_{n,0}).$$

Note V_n maps Q_n's range onto $(P_n - Q_n)$'s range and annihilates everything orthogonal to Q_n's range; V_n^* does the same with Q_n and $(P_n - Q_n)$ reversed. It is then straightforward to show that for all $i, j, k \geq 0$, $\phi^{-1}(W_{i,j})$ commutes with $(V_k + V_k^*)$, which exchanges the two "halves" of P_n's range and annihilates everything orthogonal to P_n.

Finally, define $U \equiv \sum_{n \geq 0} V_n + V_n^*$; U commutes with all the $\phi^{-1}(W_{i,j})$ and thus with all linear sums thereof. So $U \in S \cap R'$, satisfying our requirements on U. □

3 Projection lattices and their boolean completions

When H is a Hilbert space we write $\mathcal{B}(H)$ for the von Neumann algebra of all bounded linear operators on H. The set $\mathbb{P}(H)$ of all projections,

$$\mathbb{P}(H) \equiv \{P \in \mathcal{B}(H) : P^2 = P^* = P\},$$

is a complete bounded lattice with meet and join operations defined in the conventional way: $P \wedge Q$ is the projection onto the intersection of P's and Q's ranges and $P \vee Q$ is the projection onto the closed linear span of their ranges' union. (In this lattice, 1 is the identity operator and 0 the null operator.) When $\mathcal{R} \subseteq \mathcal{B}(H)$ is a von Neumann algebra, we write $\mathbb{P}_{\mathcal{R}}$ to mean its *projection lattice* $\mathbb{P}(H) \cap \mathcal{R}$; it is a complete bounded sublattice of $\mathbb{P}(H)$, as a consequence of the fact that a von Neumann algebra is closed in the strong operator topology. We write $\mathbb{P}^+_{\mathcal{R}}$ to mean $\mathbb{P}_{\mathcal{R}} \setminus \{0\}$.

Two elements P, Q of a bounded lattice are called *disjoint* if $P \wedge Q = 0$; and an element P is called disjoint from a set of elements if it is disjoint from each of them individually. Note that two projections P, Q are disjoint if and only if their ranges are disjoint aside from the null vector, and this holds irrespective of the particular von Neumann algebra $\mathcal{R}$ whose projection lattice $\mathbb{P}_{\mathcal{R}}$ we are considering P and Q to be members of.

A *partition* of an element $P \neq 0$ of a bounded lattice is a maximal pairwise-disjoint subset of the lattice's nonzero elements $\leq P$.

Definitions of $[P]_{\mathcal{S}}$ and the boolean completion $B_{\mathcal{S}}$

When $\mathcal{S}$ is a von Neumann algebra and $P \in \mathbb{P}^+_{\mathcal{S}}$, we write $[P]_{\mathcal{S}}$ to mean P's *downward closure in* $\mathbb{P}^+_{\mathcal{S}}$, i.e.

$$[P]_{\mathcal{S}} \equiv \{Q \in \mathbb{P}^+_{\mathcal{S}} : Q \leq P\};$$

and we then define $B_{\mathcal{S}}$ to be the boolean completion of $\mathbb{P}^+_{\mathcal{S}}$ realized as its regular open algebra relative to the topology with basis $\{[P]_{\mathcal{S}} : P \in \mathbb{P}^+_{\mathcal{S}}\}$. We call sets $[P]_{\mathcal{S}}$ *basis elements* of $B_{\mathcal{S}}$. For general facts about obtaining boolean completions of posets this way — including the fact that the resulting boolean algebras are always complete — see [1], Theorems 7.13 and 14.10. Concretely, $B_{\mathcal{S}}$ is the set of all subsets $X \subseteq \mathbb{P}^+_{\mathcal{S}}$ that satisfy

$$(\forall Q \in \mathbb{P}^+_{\mathcal{S}})(Q \in X \iff \text{no } Q' \leq Q \text{ in } \mathbb{P}^+_{\mathcal{S}} \text{ is disjoint from } X), \tag{†}$$

and the boolean operations on $B_{\mathcal{S}}$ are defined as follows:

$$X \wedge Y \equiv X \cap Y;$$
$$\neg X \equiv \{P : (\neg\exists Q \in X)(Q \leq P)\};$$
$$X \vee Y \equiv \neg(\neg X \wedge \neg Y).$$

The boolean ordering $\leq$ is defined as always by $X \leq Y \iff X \wedge Y = X$, and in $B_{\mathcal{S}}$ this is equivalent to $X \subseteq Y$. Since (†) ensures that every $X \in B_{\mathcal{S}}$ is downward-closed in $\mathbb{P}^+_{\mathcal{S}}$, negation can be defined equivalently as

$$\neg X \equiv \{P \in \mathbb{P}^+_{\mathcal{S}} : P \text{ disjoint from } X\}. \tag{‡}$$

Note that the mapping $P \mapsto [P]_{\mathcal{S}}$ is not a lattice embedding because $[P \vee Q]_{\mathcal{S}}$ does not equal $[P]_{\mathcal{S}} \vee [Q]_{\mathcal{S}}$ unless $P \leq Q$ or $Q \leq P$. However, $[P \wedge Q]_{\mathcal{S}} = [P]_{\mathcal{S}} \wedge [Q]_{\mathcal{S}}$ always holds.

When $\chi \subseteq \mathbb{P}^+_{\mathcal{S}}$, we write $[\chi]_{\mathcal{S}}$ to denote $\{[P]_{\mathcal{S}} : P \in \chi\}$.

Let us register a few more basic facts about $B_{\mathcal{S}}$.

Lemma 3.1. *For all $P, Q \in \mathbb{P}^+_{\mathcal{R}}$, $P \wedge Q = 0 \iff [P]_{\mathcal{R}} \wedge [Q]_{\mathcal{R}} = 0 \iff P$ is disjoint from $[Q]_{\mathcal{R}}$.* □

Corollary 3.2. *$\chi \subseteq \mathbb{P}^+_{\mathcal{R}}$ is pairwise-disjoint if and only if $[\chi]_{\mathcal{R}}$ is pairwise-disjoint.* □

Definition. A *basic partition* of $X \in B^+_{\mathcal{S}}$ is a partition of X consisting entirely of basis elements of $B^+_{\mathcal{S}}$. For subsets $\Omega \subseteq B^+_{\mathcal{S}}$, we will call $\Gamma \subseteq B^+_{\mathcal{S}}$ a "basic partition of Ω" (retaining the scare-quotes) if Γ is a basic partition of $\bigvee \Omega$ such that $\bigcup \Gamma \subseteq \bigcup \Omega$.

Lemma 3.3. *For every pairwise-disjoint subset $\chi \subseteq \mathbb{P}^+_{\mathcal{S}}$, $[\chi]_{\mathcal{S}}$ is a basic partition of $\bigvee [\chi]_{\mathcal{S}}$; conversely, every $X \in B^+_{\mathcal{S}}$ has a basic partition.*

If $\chi \subseteq \mathbb{P}^+_{\mathcal{S}}$ is pairwise-disjoint then $[\chi]_{\mathcal{S}}$ is too, by Corollary 3.2; and it is a basic fact about complete boolean algebras that any pairwise-disjoint subset is a partition of its join.

For the converse claim, fix $X \in B^+_{\mathcal{S}}$ and let χ be any maximal pairwise-disjoint subset of X; thus by the first claim $[\chi]_{\mathcal{S}}$ is a partition of *some* $B_{\mathcal{S}}$ member $Y \leq X$. If $Y < X$ strictly, there would exist $Q \in X \setminus Y$, and also by (†) a nonzero projection $Q' \in \mathbb{P}^+_{\mathcal{S}}$, $Q' \leq Q$, disjoint from Y. But $Q' \in X$ since X is downward-closed in $\mathbb{P}^+_{\mathcal{S}}$; so Q' contradicts the maximality of χ. □

Lemma 3.4. *For all $\Omega \subseteq B_{\mathcal{S}}$,*

$$\bigvee \Omega = \{Q \in \mathbb{P}^+_{\mathcal{S}} : \text{ no } Q' \leq Q \text{ in } \mathbb{P}^+_{\mathcal{S}} \text{ is disjoint from } \bigcup \Omega\}.$$

Clearly $\bigvee \Omega$ is the least $B_{\mathcal{S}}$ member that contains $\bigcup \Omega$. Suppose Q belongs to the right-hand set in the equation claimed by the lemma; i.e. no $Q' \leq Q$ in $\mathbb{P}^+_{\mathcal{S}}$ is disjoint from $\bigcup \Omega$. Since $\bigcup \Omega \subseteq \bigvee \Omega$, no $Q' \leq Q$ is disjoint from $\bigvee \Omega$ either; thus since $\bigvee \Omega$ satisfies (†) above, $Q \in \bigvee \Omega$.

Conversely, suppose some $Q \in \bigvee \Omega$ did have a $Q' \leq Q$ disjoint from $\bigcup \Omega$. By the characterization (‡) of $\neg$, we would have $\bigcup \Omega \subseteq \neg [Q']_{\mathcal{S}}$. It would follow that

$$\bigcup \Omega \subseteq \neg [Q']_{\mathcal{S}} \cap \bigvee \Omega = \neg [Q']_{\mathcal{S}} \wedge \bigvee \Omega,$$

so $\neg [Q']_{\mathcal{S}} \wedge \bigvee \Omega$ would contradict the leastness of $\bigvee \Omega$ for containing $\bigcup \Omega$. □

Lemma 3.5. *Every $\Omega \subseteq B^+_{\mathcal{S}}$ has a "basic partition."*

Fix Ω and let Γ be a maximal pairwise-disjoint set of basis elements of $B^+_{\mathcal{S}}$ such that $\bigcup \Gamma \subseteq \bigcup \Omega$. Suppose towards a contradiction that Γ is not a "basic partition of Ω", which can only happen if $\bigvee \Gamma < \bigvee \Omega$ strictly. Then there exists $Q \in \bigvee \Omega$ that is disjoint from $\bigvee \Gamma$ and hence from every member of Γ. Now $Q \in \bigvee \Omega$ entails by Lemma 3.4 that no $Q' < Q$ in $\mathbb{P}^+_{\mathcal{S}}$ is disjoint from $\bigcup \Omega$. Thus there exists $Q' < Q$ such that $Q' \in \bigcup \Omega$. But Q' is disjoint from each Γ member, so $[Q']_{\mathcal{S}}$ violates Γ's supposed maximality. □

4 Boolean embeddings corresponding to von-Neumann-algebra inclusions

In this section $\mathcal{R} \subseteq \mathcal{S}$ will be an arbitrary inclusion of von Neumann algebras. The inclusion map from $\mathbb{P}_{\mathcal{R}}$ into $\mathbb{P}_{\mathcal{S}}$ is a complete bounded-lattice embedding. Our question in this section is whether the map $[P]_\mathcal{R} \mapsto [P]_\mathcal{S}$, defined for $P \in \mathbb{P}^+_\mathcal{R}$, extends to a complete boolean embedding of $B_\mathcal{R}$ into $B_\mathcal{S}$. We begin by recalling precisely what is meant by this terminology.

A *bounded-lattice embedding* ϕ of one bounded lattice A into another B is an injective homomorphism, i.e. an injective mapping $\phi : A \to B$ such that for all $x, y \in A$,

$$\phi(1) = 1; \tag{1}$$
$$\phi(0) = 0; \tag{2}$$
$$\phi(x \vee y) = \phi(x) \vee \phi(y); \tag{3}$$
$$\phi(x \wedge y) = \phi(x) \wedge \phi(y). \tag{4}$$

If A and B are also boolean algebras, and for all $x \in A$ we have

$$\phi(\neg x) = \neg\phi(x), \tag{5}$$

then we call ϕ a *boolean embedding*. If moreover A and B are complete and we have the generalizations of (3) and (4) for all subsets $\Omega \subseteq A$, i.e.

$$(a)\ \ \phi(\bigvee \Omega) = \bigvee_{x \in \Omega} \phi(x), \qquad (b)\ \ \phi(\bigwedge \Omega) = \bigwedge_{x \in \Omega} \phi(x), \tag{6}$$

then we call ϕ a *complete boolean embedding*.

We now define a mapping ϕ with domain $B_\mathcal{R}$ which, as will show presently, is the only viable candidate to be a complete boolean embedding meeting our requirements.

$$\phi : X \mapsto \bigwedge\{Y \in B_\mathcal{S} : X \subseteq Y\}.$$

Lemma 4.1. *If $[P]_\mathcal{R} \mapsto [P]_\mathcal{S}$ extends to a complete boolean embedding $B_\mathcal{R} \to B_\mathcal{S}$, then the ϕ just defined is the unique such embedding.*

Let $\gamma : B_\mathcal{R} \to B_\mathcal{S}$ be an extending embedding as required, and let $X \in B_\mathcal{R}$ be arbitrary. Since by hypothesis $\gamma([P]_\mathcal{R}) = [P]_\mathcal{S} \ni P$ for all $P \in X$, we must have $X \subseteq \gamma(X)$. If $\gamma(X)$ were not the least $B_\mathcal{S}$-member that includes X — i.e., $\phi(X)$ — then we would have

$$\gamma(X) = \gamma(\bigvee_{P \in X} [P]_\mathcal{R}) > \bigvee_{P \in X} \gamma([P]_\mathcal{R}),$$

which would violate clause (6)(a) of the definition of complete boolean embedding. Thus $\gamma(X) = \phi(X)$, and as X was arbitrary, $\gamma = \phi$. $\square$

Lemma 4.2. *For all $X \in B_\mathcal{R}$, $\phi(X) \cap \mathbb{P}_\mathcal{R} = X$.*

It is immediate from ϕ's definition that $X \subseteq \phi(X) \cap \mathbb{P}_\mathcal{R}$, so it suffices to show that if $Q \in \phi(X) \cap \mathbb{P}_\mathcal{R}$, then $Q \in X$. Suppose Q violated this, so that by $Q \notin X$ and (†), there exists $Q' \in \mathbb{P}^+_\mathcal{R}$, $Q' \leq Q$, such that Q' is disjoint from X. Consider the $B_\mathcal{S}$ element

$\phi(X) \wedge \neg[Q']_{\mathcal{S}}$, which is strictly $< \phi(X)$. By (‡), $\neg[Q']_{\mathcal{S}}$ is the set of all $P \in \mathbb{P}^+_{\mathcal{S}}$ disjoint from Q'. Thus $X \subseteq \neg[Q']_{\mathcal{S}}$; and we have noted $X \subseteq \phi(X)$, so

$$X \subseteq \phi(X) \wedge \neg[Q']_{\mathcal{S}},$$

contradicting $\phi(X)$'s leastness in $B_{\mathcal{S}}$ for including X (its defining property). □

Corollary 4.3. *ϕ is an injective mapping.* □

Definition of The Complete Embedding Condition for $\mathcal{R} \subseteq \mathcal{S}$: For all $P \in \mathbb{P}^+_{\mathcal{R}}$, every partition of P in $\mathbb{P}^+_{\mathcal{R}}$ is also a partition of P in $\mathbb{P}^+_{\mathcal{S}}$.

Lemma 4.4. *If $\mathcal{R} \subseteq \mathcal{S}$ are von Neumann algebras, then ϕ is a complete boolean embedding $B_{\mathcal{R}} \to B_{\mathcal{S}}$ if and only if $\mathcal{R} \subseteq \mathcal{S}$ meets the Complete Embedding Condition.*

For "only if," suppose the Complete Embedding Condition does not hold, as witnessed by $P \in \mathbb{P}^+_{\mathcal{R}}$ and a partition χ of P in $\mathbb{P}^+_{\mathcal{R}}$ which is not a partition of P in $\mathbb{P}^+_{\mathcal{S}}$. This can only be because χ fails to be maximal, as a set of pairwise-disjoint projections $\leq P$ in $\mathbb{P}^+_{\mathcal{S}}$. Let $Q < P$ witness this failure of maximality. Since Q is disjoint from χ,

$$[Q]_{\mathcal{S}} \not\leq \bigvee_{P' \in \chi} [P']_{\mathcal{S}} = \bigvee_{P' \in \chi} \phi([P']_{\mathcal{R}}),$$

but

$$[Q]_{\mathcal{S}} < [P]_{\mathcal{S}} = \phi([P]_{\mathcal{R}}) = \phi(\bigvee [\chi]_{\mathcal{R}}),$$

implying $\phi(\bigvee[\chi]_{\mathcal{R}}) \neq \bigvee_{P' \in \chi} \phi([P']_{\mathcal{R}})$, which violates clause (6)(a) of complete embeddings.

For the "if" direction, let $\mathcal{R} \subseteq \mathcal{S}$ meet the Complete Embedding Condition. We have seen (Corollary 4.3) that ϕ is injective; we must verify that conditions (1)-(6) given above always hold for ϕ. (1) and (2) are trivial. (3) and (4) are special cases of (6)(a) and (6)(b) respectively, so we need not prove them separately.

Let us show that (6)(a) holds. Let $\Omega \subseteq B_{\mathcal{R}}$ be arbitrary. We observe first that $\bigvee_{X \in \Omega} \phi(X)$ is the least $B_{\mathcal{S}}$ member containing $\bigcup \Omega$, and that $\phi(\bigvee \Omega)$ is the least $B_{\mathcal{S}}$ member containing the least $B_{\mathcal{R}}$ member containing $\bigcup \Omega$. Thus we have

$$\bigvee_{X \in \Omega} \phi(X) \leq \phi(\bigvee \Omega).$$

Suppose now that Ω is a counterexample to (6)(a); the above inequality must then be strict:

$$\bigvee_{X \in \Omega} \phi(X) < \phi(\bigvee \Omega).$$

By Lemma 3.5 there exists a "basic partition" Γ of Ω. By the definition of "basic partition" we have $\bigvee \Gamma = \bigvee \Omega$ and so $\phi(\bigvee \Gamma) = \phi(\bigvee \Omega)$. Also by that definition we have $\bigcup \Gamma \subseteq \bigcup \Omega$; and since $\bigvee_{X \in \Gamma} \phi(X)$ is the least $B_{\mathcal{S}}$ member containing $\bigcup \Gamma$, we have

$$\bigvee_{X \in \Gamma} \phi(X) \leq \bigvee_{X \in \Omega} \phi(X).$$

All this is to say that, under the supposition that there exists a counterexample to (6)(a), we have a pairwise-disjoint set Γ of basis elements of $B_{\mathcal{R}}$ such that

$$\bigvee_{X \in \Gamma} \phi(X) < \phi(\bigvee \Gamma). \tag{$*$}$$

Next, we show there must exist some $P \in \mathbb{P}^+_{\mathcal{R}}$ such that $P \in \phi(\bigvee \Gamma) \setminus \bigvee_{X \in \Gamma} \phi(X)$. Suppose not, which by $(*)$ implies that $\bigvee_{X \in \Gamma} \phi(X)$ and $\phi(\bigvee \Gamma)$ have the same intersection with $\mathbb{P}^+_{\mathcal{R}}$. By Lemma 4.2, $\phi(\bigvee \Gamma) \cap \mathbb{P}^+_{\mathcal{R}} = \bigvee \Gamma$. Thus $\bigvee_{X \in \Gamma} \phi(X)$ would include $\bigvee \Gamma$; but $\phi(\bigvee \Gamma)$ is by definition the least $B_{\mathcal{S}}$ element that includes $\bigvee \Gamma$, so we would have $\bigvee_{X \in \Gamma} \phi(X) \geq \phi(\bigvee \Gamma)$, violating $(*)$.

Now fix a $P \in \mathbb{P}^+_{\mathcal{R}}$ as in the previous paragraph and define $\chi \equiv \{Q \wedge P : [Q]_{\mathcal{R}} \in \Gamma\}$. It is easily verified that χ is a partition of P in $\mathbb{P}_{\mathcal{R}}$. By the Complete Embedding Condition, χ is also a partition of P in $\mathbb{P}_{\mathcal{S}}$. Thus every $Q \in \mathbb{P}^+_{\mathcal{S}}$ that is $\leq P$ meets some member of χ, and so fails to be disjoint from $\bigcup \Gamma$. Since $\bigcup \Gamma \subseteq \bigvee_{X \in \Gamma} \phi(X)$, every such Q would fail to be disjoint from $\bigvee_{X \in \Gamma} \phi(X)$; so by (†), $P \in \bigvee_{X \in \Gamma} \phi(X)$, contradicting our choice of P.

Thus we have shown that (6)(a) has no counterexamples. Our next task is to show that (5) doesn't either, i.e. that for arbitrary $X \in B_{\mathcal{R}}$ we have $\neg\phi(X) = \phi(\neg X)$. Observe that $\neg\phi(X)$ is the unique $B_{\mathcal{S}}$ member whose join with $\phi(X)$ is 1 and whose meet with $\phi(X)$ is 0; thus our task is equivalent to proving that $\phi(\neg X)$ meets these two criteria. Plugging in $\{X, \neg X\}$ for Ω in (6)(a) we have:

$$\phi(\bigvee \{X, \neg X\}) = \phi(X) \vee \phi(\neg X).$$

The left-hand side of this equation is $\phi(1)$, i.e. 1, so the join requirement is met. For the meet requirement, suppose it failed, so that $\phi(\neg X) \wedge \phi(X) > 0$. Let Q be any projection in $\phi(\neg X) \wedge \phi(X)$. Q cannot be disjoint from X, lest $\phi(X) \wedge \neg[Q]_{\mathcal{S}}$ be a $B_{\mathcal{S}}$ member that contains X but is smaller than $\phi(X)$, contradicting the latter's leastness for this property. Thus there exists $Q' \leq Q$, $Q' \in \mathbb{P}^+_{\mathcal{S}}$, such that $Q' \in \phi(\neg X)$ and $Q' \leq$ some projection in X. Now by (‡), $\neg X$ is the set of $\mathbb{P}^+_{\mathcal{R}}$ members disjoint from X, and $\phi(\neg X)$ is the least $B_{\mathcal{S}}$ member containing it. But our Q' contradicts this: since supposedly $Q' \in \phi(\neg X)$, $\phi(\neg X) \wedge \neg[Q']_{\mathcal{S}}$ would be a strictly smaller $B_{\mathcal{S}}$ element, and it would still contain $\neg X$ because Q' is $\leq$ some projection in X.

To confirm that (6)(b) holds, note that the $\neg$ operation is an order-reversing involution (in both $B_{\mathcal{R}}$ and $B_{\mathcal{S}}$), and ϕ preserves $\neg$, so if Ω were a counterexample to (6)(b), then $\{\neg X : X \in \Omega\}$ would be a counterexample to (6)(a); but we have shown that no such counterexample exists. □

5 From rigid *-algebraic inclusions to rigid boolean inclusions

In this section we propose a way of deriving a rigid boolean inclusion, meeting the requirements of the Main Question above, from a rigid *-algebraic inclusion of von Neumann algebras; and we isolate the conjecture on which its success depends. Our tools are the boolean completions $B_{\mathcal{R}}$ and $B_{\mathcal{S}}$ from Section 3, and the canonical mapping $\phi : B_{\mathcal{R}} \to B_{\mathcal{S}}$ from Section 4.

Definition of $B_{\mathcal{R}\subseteq\mathcal{S}}$. Let $B_{\mathcal{R}\subseteq\mathcal{S}}$ be the image of $B_{\mathcal{R}}$ under the ϕ defined above.

The Main Conjecture: If $\mathcal{R} \subset \mathcal{S}$ is a rigid *-algebraic inclusion of distinct von Neumann factors, with $\mathcal{S}$ type III, then

(a) $\mathcal{R} \subset \mathcal{S}$ meets the Complete Embedding Condition (defined in Section 4);

(b) If there is a nontrivial boolean automorphism of $B_{\mathcal{S}}$ that leaves each $B_{\mathcal{R}\subseteq\mathcal{S}}$-member fixed then there is a nontrivial lattice automorphism of $\mathbb{P}_{\mathcal{S}}$ that leaves each $\mathbb{P}_{\mathcal{R}}$-member fixed.

Theorem. *If the Main Conjecture holds, and $\mathcal{R} \subseteq \mathcal{S}$ is a simple inclusion of distinct injective von Neumann factors with $\mathcal{S}$ type III (as is possible by the Simple Subfactor Theorem stated in Section 2) then $B_{\mathcal{R}\subseteq\mathcal{S}} \subseteq B_{\mathcal{S}}$ is a rigid complete boolean inclusion of complete, nonrigid, atomless boolean algebras, providing an affirmative answer to our Main Question.*

Fix $\mathcal{R} \subseteq \mathcal{S}$ as in the theorem's statement. By clause (a) of the Main Conjecture and Lemma 4.4, $B_{\mathcal{R}\subseteq\mathcal{S}}$ is a complete boolean subalgebra of $B_{\mathcal{S}}$. $B_{\mathcal{S}}$ is atomless because $\mathcal{S}$, being non-type-I, has no minimal projections; the same is true for $B_{\mathcal{R}\subseteq\mathcal{S}}$ since by Lemma 2.1 $\mathcal{R}$ is not type I either. Every nontrivial von Neumann algebra has a nontrivial *-automorphism, whose restriction to projections is a nontrivial lattice automorphism; it follows that neither $\mathbb{P}_{\mathcal{R}}$ nor $\mathbb{P}_{\mathcal{S}}$ is a rigid lattice, and thence that neither $B_{\mathcal{R}\subseteq\mathcal{S}}$ nor $B_{\mathcal{S}}$ is a rigid boolean algebra. It remains to show the rigidity of the inclusion; therefore by clause (b) of the Main Conjecture it suffices to show that $\mathbb{P}_{\mathcal{R}} \subseteq \mathbb{P}_{\mathcal{S}}$ is a rigid lattice inclusion.

Lemma 5.1. *With $\mathcal{R} \subseteq \mathcal{S}$ meeting the hypothesis of the Theorem above, $\mathbb{P}_{\mathcal{R}} \subseteq \mathbb{P}_{\mathcal{S}}$ is a rigid lattice inclusion.*

This is a consequence of results in [4]. That paper makes essential use of the *-algebra $LS(M) \supseteq M$ of *locally measurable operators* relative to a von Neumann algebra M. Because our factors are type III, and because, as noted on p. 5 of [4], $LS(M) = M$ for such factors, we will not bother defining $LS(M)$ here. We will also adopt from [4] the notation $l(X)$ to mean the left support projection of a Hilbert-space operator X, i.e. the projection onto the closure of X's range. Note also that by a "real *-automorphism" ψ of a von Neumann algebra is meant a ring automorphism that preserves the adjoint relation (i.e. $\psi(T^*) = \psi(T)^*$) and is real-linear (i.e. $\psi(rT) = r\psi(T)$ for real r).

In order to prove Lemma 5.1 we will establish some sub-lemmas, beginning with the following immediate corollary of Theorems A and B from [4]:

Fact 5.2 (Corollary of [4], Theorems A and B)**.** *Suppose that M is a type III von Neumann algebra, and that $\Phi : \mathbb{P}_M \to \mathbb{P}_M$ is a lattice automorphism; then there exist a real *-automorphism ψ of M and an invertible element $Y \in LS(M)$ such that for all $X \in LS(M)$, $\Phi(l(X)) = l(Y\psi(X)Y^{-1})$.* $\square$

The next lemma applies Fact 5.2 to our special case, where M is a factor.

Lemma 5.3. *Suppose that M is a type III von Neumann factor, and that $\Phi : \mathbb{P}_M \to \mathbb{P}_M$ is a lattice automorphism; then there exist a (complex-linear) *-automorphism ψ of M and an invertible element $Y \in M$ such that for all $P \in \mathbb{P}_M$, $\Phi(P) = l(Y\psi(P)Y^{-1})$.*

Note first that "$LS(M)$" can be replaced in Fact 5.2 with "M" because we are assuming M is a factor (see p.5 of [4]). Second, we consider only projections $P \in LS(M)$ rather than arbitrary operators $X \in LS(M)$, which allows us to replace the equation $\Phi(l(X)) = l(Y\psi(X)Y^{-1})$ with $\Phi(P) = l(Y\psi(P)Y^{-1})$.

Finally, we justify the replacement of "real *-automorphism" with "(complex-linear) *-automorphism" as follows. Clause (4) of [4]'s Lemma 2.1 entails that when M is a factor, so that 0 and 1 are its only central projections, a real *-automorphism ψ of M must be either a (complex-linear) *-automorphism or a conjugate-linear *-automorphism. Suppose we have an automorphism ψ of the latter type, which along with an operator Y satisfies the lemma's statement. Then the mapping ψ^* defined by $\psi^*(T) \equiv \psi(T^*)$ is an automorphism of M of the former type, since $\psi^*(cT) = \psi((cT)^*) = \psi(\overline{c}T) = c\psi(T)$, where c is a complex number and $\overline{c}$ its complex conjugate. And $\Phi(P) = Y\psi^*(P)Y^{-1}$ still holds for all projections P since $P^* = P$. □

Lemma 5.4. *If $A \in \mathcal{B}(H)$ is invertible and $(\forall P \in \mathbb{P}_{\mathcal{R}})(P = l(APA^{-1}))$, then $A \in \mathcal{R}'$.*

Fix $P \in \mathbb{P}_{\mathcal{R}}$. Since A is invertible we have range(PA) = range(PA^{-1}) = range(P). This along with the lemma's hypothesis ensures that APA^{-1} maps P's range into itself, and (applying the hypothesis to $(1-P)$) that $A(1-P)A^{-1}$ maps $(1-P)$'s range into itself. Since an operator A commutes with a projection P if and only if A maps P's range into itself, and $(1-P)$'s range into itself, A commutes with P. As P was arbitrary, A commutes with all projections in $\mathcal{R}$. It then follows from the way von Neumann algebras are generated by their projections (i.e. the spectral theorem; see [2], Theorem 5.2.2) that A commutes with all of $\mathcal{R}$, i.e. $A \in \mathcal{R}'$. □

Proof of Lemma 5.1.

Recall that we are supposing in the present section that $\mathcal{R} \subset \mathcal{S}$ is a simple, hence rigid, *-algebra inclusion of type III factors. Suppose towards a contradiction of Lemma 5.1's claim that there exists a nontrivial lattice automorphism Φ of $\mathbb{P}_{\mathcal{S}}$ that acts trivially on $\mathbb{P}_{\mathcal{R}}$. Then by Lemma 5.3 there exist a complex-linear *-automorphism ψ of $\mathcal{S}$ and an invertible $Y \in \mathcal{S}$ such that for all $P \in \mathbb{P}_{\mathcal{S}}$, $\Phi(P) = l(Y\psi(P)Y^{-1})$.

Since Φ supposedly fixes each $\mathbb{P}_{\mathcal{R}}$-member, we have

$$(\forall P \in \mathbb{P}_{\mathcal{R}})(P = l(Y\psi(P)Y^{-1})).$$

By polar decomposition ([2], Prop. 6.1.3), $Y = TU$ for some (unique) self-adjoint $T \in \mathcal{S}$ and unitary $U \in \mathcal{S}$. (In general the U component of a polar decomposition is a partial isometry but for invertible Y, U must furthermore be unitary, and T must be invertible.)

By the unitary implementation theorem ([2], Theorem 7.2.9), there exists a unitary operator $V \in \mathcal{B}(H)$ such that $(\forall T \in \mathcal{S})(\psi(T) = VTV^{-1})$. Thus for all $P \in \mathbb{P}_{\mathcal{R}}$,

$$P = l(TUVPV^{-1}U^{-1}T^{-1}).$$

Invoking Lemma 5.4 with A set to (TUV), we have $(TUV) \in \mathcal{R}'$. Both components of (TUV)'s polar decomposition must also belong to $\mathcal{R}'$ (since $\mathcal{R}'$ is a von Neumann algebra), and these components are just T and UV. So $T \in \mathcal{R}'$ and $UV \in \mathcal{R}'$. Since also $T \in \mathcal{S}$, and $\mathcal{R}$ (being a simple subfactor of $\mathcal{S}$) has trivial relative commutant in $\mathcal{S}$, T must be trivial, i.e. a scalar multiple of 1.

Now since the unitary V has been defined from the *-automorphism of $\mathcal{S}$ that it induces, and the unitary U belongs to $\mathcal{S}$, UV is unitary and induces a *-automorphism of $\mathcal{S}$. This *-automorphism cannot act trivially on $\mathcal{S}$; otherwise, since T is trivial (of form $c1$), the lattice automorphism

$$\Phi : P \mapsto l(TUVPV^{-1}U^{-1}T^{-1})$$

of $\mathbb{P}_{\mathcal{S}}$ would be trivial, violating our supposition on Φ.

Thus UV induces a nontrivial *-automorphism of $\mathcal{S}$; but since $UV \in \mathcal{R}'$ it acts trivially on $\mathcal{R}$, contradicting the *-algebraic rigidity of $\mathcal{R} \subseteq \mathcal{S}$. $\square$

References

[1] T. Jech, *Set Theory*, 3rd Edition, Springer, 2003.

[2] R. Kadison and J. Ringrose, *Fundamentals of the Theory of Operator Algebras*, AMS, 1997.

[3] R. Longo, “Simple injective subfactors.” Adv. in Math., 63(2):152–171, 1987.

[4] M. Mori, “Lattice isomorphisms between projection lattices of von Neumann algebras,” arXiv