Noncommutative analogs of random-real forcing

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Abstract

The measure algebra used in random-real forcing is isomorphic (as is well known) to the projection lattice of a commutative von Neumann algebra $\mathcal{R}$, and it is not difficult to show that a generic filter G on this lattice induces a normal state ω_G on $\mathcal{R}$, with which it is interdefinable. We show that this construction also works when carried out with noncommutative von Neumann algebras, inducing generic states on them that can be seen as noncommutative analogs of random reals. We consider whether these forcing notions can or must collapse $2^{\aleph_0}$.

1 Introduction / Overview

All homogeneous c.c.c. measure algebras are isomorphic, by Maharam's Theorem. Two well-known instances of this algebra are, first, the algebra B of equivalence classes of Borel subsets of $\mathbb{R}$ modulo Lebesgue-null sets, and, second, the projection lattice $\mathbb{P}$ of a commutative type II von Neumann algebra $\mathcal{R}$ acting on a separable Hilbert space H.

Forcing with B is known as *random-real forcing*: the members of a generic filter on B are subsets (modulo null sets) of an underlying space, namely $\mathbb{R}$, that converge to a generic limit point of that space, namely to the random real number x_G. See Chapter 15 of [3] for a precise account.

Since B and $\mathbb{P}$ are order-isomorphic they are equivalent as forcing notions, so one might expect a similar convergence to a generic limit point of an underlying space to occur when $\mathbb{P}$ is used for forcing. This is indeed the case, but only if the projections in $\mathbb{P}$ are construed as subsets of an appropriate underlying space.

If each $P \in \mathbb{P}$ is identified as usual with a subset of the Hilbert space H, namely with its own range space, then a generic filter of these subsets will *not* converge to any generic limit point or limit ray of H. (See the Appendix, Section 7 for a precise statement and proof of this claim.) But we can instead identify each P with a subset of $\mathcal{R}$'s state space, namely with the set of states on $\mathcal{R}$ induced by the vectors in P's range. Alternatively we can focus just on these states' restrictions to $\mathbb{P}$, which are known as *probability measures*: a unit vector $v \in H$ induces the probability measure $\mu_v : Q \mapsto \|Qv\|^2$ on $\mathbb{P}$. For a fixed projection $Q \in \mathbb{P}$, the possible values of $\|Qv\|^2$ can be confined into narrower intervals as the unit vector v is constrained into the ranges of smaller P. We will verify (Theorem 3.4) that for each Q, the members of a generic filter on $\mathbb{P}$ narrow these intervals down to a single point $x_Q \in \{0,1\}$, and that the resulting function $Q \mapsto x_Q$ is a (two-valued) generic probability measure $\mu_G : \mathbb{P} \to [0,1]$. In this way, relative to a topology of pointwise convergence, the sets of probability measures identified with G's members converge to μ_G.

Our main goal is to show that this same construction can also be carried out with noncommutative von Neumann algebras $\mathcal{R}$. Proving that generic probability measures μ_G are still induced in these cases is more complicated; we do so in Section 4 with arguments involving spectral resolutions. These μ_G can be considered noncommutative analogs of random real numbers; unlike the μ_G from the commutative case, these μ_G can take values strictly between 0 and 1.

In Section 5 we confirm that μ_G extends to a normal state ω_G on $\mathcal{R}$ with which it is interdefinable. Section 6 registers some salient facts and questions about forcing with noncommutative $\mathbb{P}$. We ask notably whether it can or must collapse uncountable cardinals, and we offer some facts that may be useful for answering this question.

2 Preliminary definitions

M is a ground model of ZFC in which all the following objects are defined.

H is a complex Hilbert space of countably infinite dimension.

$\mathcal{B}(H)$ is the algebra of all bounded operators acting on H.

$\mathcal{R} \subseteq \mathcal{B}(H)$ is a von Neumann algebra acting on H having no minimal (non-null) projections.

Remark. We assume familiarity with von Neumann algebras, referring to [4] for basic results. We do not in general assume that $\mathcal{R}$ is commutative.

$\mathbb{P} \subseteq \mathcal{R}$ is the lattice of all (orthogonal) projection operators in $\mathcal{R}$ with the usual ordering

$$P \leq Q \iff \operatorname{ran}(P) \subseteq \operatorname{ran}(Q).$$

$\mathbb{P}^+$ is $\mathbb{P}$ without the null projection 0. We use 1 to denote the identity projection. The symbols $\wedge$ and $\vee$ denote the usual lattice meet and join operations on $\mathbb{P}$.

We write $\operatorname{ran}^{=1}(P)$ as shorthand for the set of unit vectors in P's range.

A *probability measure* on a set X of projections satisfying $1 \in X$ (such as $\mathbb{P}$) is a real function $\mu : X \to [0,1]$ that satisfies $\mu(1) = 1$ and is additive in the sense that it satisfies

$$\mu(\bigvee_{P \in Y} P) = \sum_{P \in Y} \mu(P)$$

for every pairwise-orthogonal subset $Y \subseteq X$, finite or countably infinite, whose join exists in X.

For a unit vector $v \in H$, $\mu_v : \mathbb{P} \to [0,1]$ is the probability measure $Q \mapsto \|Qv\|^2$ that v induces.

For $P \in \mathbb{P}^+$ and $Q \in \mathbb{P}$, $I_P(Q) \subseteq [0,1]$ is the smallest closed interval containing all values of $\mu_v(Q)$ for $v \in \mathrm{ran}^{=1}(P)$, that is, the closed interval such that

$$\max(I_P(Q)) = \sup\{\|Qv\|^2 : v \in \mathrm{ran}^{=1}(P)\};$$

$$\min(I_P(Q)) = \inf\{\|Qv\|^2 : v \in \mathrm{ran}^{=1}(P)\}.$$

We write $|I|$ for $\max(I) - \min(I)$.

Remark. A projection is sometimes said to be *isoclinic* to another if every vector in the former's range space lies at the same angle to the latter's range space. $|I_P(Q)|$ can be thought of geometrically as the degree to which P diverges from isoclinicity to Q; when $|I_P(Q)| = 0$, P is isoclinic to Q. Geometrically, then, the goal of the next two sections will be to show, for arbitrary fixed $Q \in \mathbb{P}^+$, that the projections in a generic filter $G \subseteq \mathbb{P}^+$ approach (or attain) isoclinicity to Q.

3 Inducing generic probability measures

We now consider forcing with $\mathbb{P}$. We stress that we are not assuming $\mathcal{R}$ to be commutative. Let G be a filter on $\mathbb{P}^+$ that is generic with respect to M, meaning $G \cap D \neq \emptyset$ for all dense subsets $D \subseteq \mathbb{P}^+$ that exist in M. $M[G]$ is the corresponding generic extension. Our stipulation that $\mathbb{P}^+$ has no minimal projections ensures that the forcing is nontrivial, *i.e.* $M[G] \neq M$.

Fact 3.1 $P \leq P' \Rightarrow I_P(Q) \subseteq I_{P'}(Q)$. $\square$

Fact 3.1 is obvious but worth pointing out because $I_P(Q)$ will be the interval into which P, considered as a forcing condition, confines the value that the induced generic probability measure will take on Q; smaller conditions will confine the value into smaller intervals.

Definition. $\overline{I_P(Q)}$ is the closure of $I_P(Q)$ in $M[G]$'s (potentially "thicker") real line.

Lemma 3.2 *Let $Q \in \mathbb{P}$ be arbitrary; then*

$$\bigcap_{P \in G} \overline{I_P(Q)} = \{x\}$$

will hold for some real $x \in M[G]$, if and only if for all $\epsilon > 0$ there exists $P \in G$ such that $|I_P(Q)| < \epsilon$.

Suppose $P \in G$ exists as demanded for all $\epsilon > 0$. For all $P, P' \in G$, $\overline{I_P(Q)} \cap \overline{I_{P'}(Q)} \neq \emptyset$, thanks to the mutual compatibility of G-members; then by the compactness of the real interval $[0,1]$ in $M[G]$, the intersection of $\overline{I_P(Q)}$ for all $P \in G$ must have at least one point. And it can have at most one point because the $\overline{I_P(Q)}$'s have arbitrarily small length.

Conversely, suppose the intersection is $\{x\}$ for some real $x \in M[G]$, and fix $\epsilon > 0$. Plainly we have $\max(I_P(Q)) \geq x$ whenever $P \in G$, as well as $\min(I_P(Q)) \leq x$. Thus there must exist some $P' \in G$ such that

$$x \leq \max(I_{P'}(Q)) \leq x + \epsilon/2, \tag{*}$$

and some $P'' \in G$ such that

$$x - \epsilon/2 \leq \min(I_{P''}(Q)) \leq x. \tag{**}$$

By pairwise compatibility of filter elements, $P' \wedge P'' \in G$; and $P' \wedge P''$ serves as the required P, since by Fact 3.1 $I_{P' \wedge P''}(Q)$ satisfies both (*) and (**). $\square$

Definition of induced probability measure. When Lemma 3.2's condition holds for all $Q \in \mathbb{P}$, the following function $\mu_G : \mathbb{P} \to [0, 1]$ is well defined in $M[G]$:

$$\mu_G : Q \mapsto \text{ the sole member of } \bigcap_{P \in G} \overline{I_P(Q)}.$$

We call μ_G the *generic function induced on* $\mathbb{P}$ *by* G. If μ_G is a probability measure, we call it the *generic probability measure induced by* G.

Lemma 3.3 *G is forced by all conditions to induce a generic probability measure on $\mathbb{P}$ if and only if, for all $P \in \mathbb{P}^+$, $\epsilon > 0$, and sets $\{Q_i : i < \alpha\} \subseteq \mathbb{P}$ of pairwise-orthogonal projections where $0 < \alpha \leq \omega_0$, there exist $n > 0$ and $P' \in \mathbb{P}^+$, $P' \leq P$, such that, when we define $Q_{<n} \equiv \sum_{i<n} Q_i$ and $Q_{<\alpha} \equiv \sum_{i<\alpha} Q_i$, we have*

$$\min(I_{P'}(Q_{<n})) > \max(I_{P'}(Q_{<\alpha})) - \epsilon.$$

Suppose all $P \in \mathbb{P}^+$ force G to induce a generic probability measure μ_G on $\mathbb{P}$. Fix G (which fixes μ_G) and consider arbitrary $P \in G$, $\{Q_i : i < \alpha\}$, and ϵ as in the lemma's statement. Since μ_G is countably additive we may choose $n \in \mathbb{N}$ sufficiently large so that

$$\mu_G(Q_{<\alpha}) \ \geq \ \mu_G(Q_{<n}) \ > \ \mu_G(Q_{<\alpha}) - \epsilon/3, \tag{†}$$

when $Q_{<n}$ and $Q_{<\alpha}$ are defined as in the statement of the lemma. (If α is finite then simply let n equal α; in this case, $Q_{<n} = Q_{<\alpha}$.) We now invoke Lemma 3.2 twice in succession to obtain some $P' \in G$ such that

$$|I_{P'}(Q_{<\alpha})| < \epsilon/3; \quad |I_{P'}(Q_{<n})| < \epsilon/3. \tag{††}$$

Since $P' \in G$, the definition of induced probability measure implies $\mu_G(Q_{<n}) \in \overline{I_{P'}(Q_{<n})}$ and $\mu_G(Q_{<\alpha}) \in \overline{I_{P'}(Q_{<\alpha})}$. It follows from this and from (†), (††) that n and P' are as required.

Conversely, suppose that for all P, $\{Q_i : i < \alpha\}$ and ϵ as in the lemma's statement, there exist the required $n > 0$ and $P' \leq P$. Note first that this holds in particular for all singletons $\{Q_0\}$; it follows easily that the condition of Lemma 3.2 is met for all $Q \in \mathbb{P}$, meaning that a generic *function* μ_G on $\mathbb{P}$ is induced by G. Clearly $\mu_G(1) = 1$. It remains to verify μ_G's countable additivity. Let $\dot{\mu_G}$ be a forcing name for μ_G. Fix arbitrary P, ϵ and pairwise-orthogonal $\{Q_i : i < \alpha\} \subseteq \mathbb{P}$. For all $P' \leq P$, the condition in the lemma's statement gives us $n > 0$ and $P'' \leq P'$ such that

$$P'' \Vdash \dot{\mu_G}(Q_{<n}) > \dot{\mu_G}(Q_{<\alpha}) - \epsilon.$$

Since such n and P'' exist for all $P' \leq P$ and arbitrarily small ϵ,

$$P \Vdash \lim_{n\to\infty} \dot{\mu_G}(Q_{<n}) = \dot{\mu_G}(Q_{<\alpha}). \quad \Box$$

Theorem 3.4 (The commutative case) *If $\mathbb{P}$ is the projection lattice of a commutative von Neumann algebra $\mathcal{R}$, then a generic filter G on $\mathbb{P}^+$ induces a generic probability measure μ_G.*

This claim will follow from the more general Theorem 4.10, but there is a simpler proof available in the present case, where $\mathcal{R}$ is commutative. This proof will also show that in the present case μ_G takes only values 0 and 1.

Preliminary observation: if $P \leq Q$ then $I_P(Q)$ is the singleton interval $[1, 1]$; and if P is orthogonal to Q then $I_P(Q) = [0, 0]$.

It suffices to show that the conditions of Lemma 3.3 are met by every $P \in \mathbb{P}^+$. Fix P; then, as in Lemma 3.3, fix a set $\{Q_i : i < \alpha\} \subseteq \mathbb{P}$ of pairwise-orthogonal projections, where $0 < \alpha \leq \omega_0$, and let $Q_{<\alpha}$ denote their sum. Fix $\epsilon > 0$ too (although as we will see the value does not enter into the argument).

For all $i < \alpha$, P commutes with Q_i. It follows from the geometry of commuting projections that either P and Q_i are orthogonal, or their meet $P \wedge Q_i$ (which is itself a $\mathbb{P}$-member) is non-null. If every Q_i is orthogonal to P (Case I), let P' just be P, and set $n = 1$. If instead $P \wedge Q_i \neq 0$ for some i (Case II), let P' be $P \wedge Q_i$ for the least such i, and set $n = i + 1$ for this i. Note that in both cases $P' \leq P$.

Write $Q_{<n}$ for $\sum_{i<n} Q_i$, as in Lemma 3.3.

In Case I, $I_{P'}(Q_i) = [0, 0]$ (by our preliminary observation above) for all i, and therefore

$$\min(I_{P'}(Q_{<n})) = \max(I_{P'}(Q_{<\alpha})) = 0.$$

In Case II, $I_{P'}(Q_i) = [0, 0]$ except for one i, $i < n$, for which $I_{P'}(Q_i) = [1, 1]$; thus

$$\min(I_{P'}(Q_{<n})) = \max(I_{P'}(Q_{<\alpha})) = 1.$$

In either case the requirement of Lemma 3.3 is met, regardless of ϵ's value. $\Box$

4 Noncommutative analogs of random real forcing

In this section we show that forcing with $\mathbb{P}^+$ induces a generic probability measure on $\mathbb{P}$ when $\mathbb{P}$ is defined from any $\mathcal{R}$ without minimal projections, and not just when $\mathcal{R}$ is commutative. The basic idea will be to take QP's polar decomposition, and then a sufficiently thin "slice" of the spectral resolution of the self-adjoint part. We begin by recalling the definitions of these terms.

Definition. For $A \in \mathcal{B}(H)$, the *polar decomposition* of A is a pair U, T of operators such that $A = UT$, with T a non-negative self-adjoint operator, and U a partial isometry whose initial space is the closure of T's range.

Fact 4.1 ([4], Proposition 6.1.3) *For all $A \in \mathcal{R}$, a polar decomposition $A = UT$ exists, with $U, T \in \mathcal{R}$.* $\square$

Definition. The *spectrum* $sp(T)$ of an operator T is defined by

$$sp(T) \equiv \{\lambda \in \mathbb{C} : (T - \lambda 1) \text{ is not an invertible operator}\},$$

where 1 denotes the identity projection. For self-adjoint $T \in \mathcal{R}$, a *spectral resolution of* T is a set $\{E_\lambda : \lambda \in \mathbb{R}\}$ of projections monotonically increasing with λ, with $E_{\|T\|} = 1$, such that T is the norm limit of a sequence of *spectral approximations* A of form

$$A = \sum_{j=1}^{n} \lambda_j (E_{\lambda_j} - E_{\lambda_{j-1}}),$$

where the λ_j increase with j, and $\lambda_0 < -\|T\|$, and $\lambda_n = \|T\|$, and $(E_{\lambda_j} - E_{\lambda_{j-1}}) = 0$ if and only if $sp(T) \cap (\lambda_{j-1}, \lambda_j] = \emptyset$.

Lemma 4.2 (Spectral approximation theorem) *For all self-adjoint $T \in \mathcal{R}$, a spectral resolution exists, is unique, and is a subset of $\mathbb{P}$.*

This follows from [4], Theorems 5.2.2 and 5.2.3. $\square$

We now establish several preliminary lemmas, leading up to Theorem 4.10. For all of these lemmas, assume that P, Q are arbitrary projections in $\mathbb{P}^+$, and that $UT = QP$ is the polar decomposition of QP.

Lemma 4.3 *For all $v \in \mathrm{ran}(P)$, $\|Qv\| = \|Tv\|$.*

Tv is in T's range, whose closure is the initial space of partial isometry U, by the definition of polar decomposition. Partial isometries are norm-preserving on their initial spaces, so for all vectors v, $\|Tv\| = \|UTv\| = \|QPv\|$. And if $v \in \mathrm{ran}(P)$ then $\|QPv\| = \|Qv\|$. $\square$

Definition. The *support projection* $\sigma(T)$ of T is the projection onto $\ker^\perp(T)$, that is, onto the orthogonal complement of T's null space. Note that $T \in \mathcal{R} \Rightarrow \sigma(T) \in \mathcal{R}$.

Lemma 4.4 $\sigma(T) \leq P$.

It is easily verified that

$$Pv = 0 \Rightarrow QPv = 0 \Rightarrow UTv = 0 \Rightarrow Tv = 0,$$

which entails $\ker(P) \subseteq \ker(T)$, and therefore $\ker^{\perp}(T) \subseteq \ker^{\perp}(P)$. It follows from this, and the fact that the projection onto $\ker^{\perp}(P)$ is just P, that $\sigma(T) \leq P$. □

Lemma 4.5 *If $\{E_\lambda : \lambda \in \mathbb{R}\}$ is T's spectral resolution, then $(E_{\|T\|} - E_0) = \sigma(T)$.*

T is non-negative by the definition of polar decomposition given above, so its spectrum $sp(T)$ is a subset of the interval $[0, \|T\|]$. It follows from this and the definition of spectral resolution that $E_\lambda = 0$ for all $\lambda < 0$, and that $\mathrm{ran}(E_0) = \ker(T)$. Thus

$$\mathrm{ran}(1 - E_0) = \ker^{\perp}(T) = \mathrm{ran}(\sigma(T)),$$

which along with the fact that $E_{\|T\|} = 1$ entails $(E_{\|T\|} - E_0) = \sigma(T)$. □

Lemma 4.6 *T is the norm limit of a sequence of spectral approximations to T that are determined by finite sequences of λ_j's satisfying $\lambda_1 = 0$.*

Suppose $\lambda_0, \ldots, \lambda_n$ determine a spectral approximation A to T. If 0 is not among the λ_i, we can add it as another λ_i and re-index, yielding a spectral approximation A' at least as close to T in norm as A. Since T is a non-negative self-adjoint operator, we have $E_\gamma = 0$ for all $\gamma < 0$, so we can throw out all $\lambda_j < 0$ except for one, re-index the remaining ones again if necessary; this sequence will determine the same A' and we will have $\lambda_1 = 0$. Thus if some sequence of spectral approximations converges to T, so does some sequence all of whose members meet the requirement stated in the lemma. □.

Lemma 4.7 *If $\{E_\lambda : \lambda \in \mathbb{R}\}$ is T's spectral resolution, and $\lambda_0, \ldots, \lambda_n (n > 2)$ determine a spectral approximation A to T with $\lambda_1 = 0$, and $v \in \mathrm{ran}^{=1}(\sigma(T))$, then some j satisfying $0 < \lambda_j \leq \|Av\|$ must also satisfy $(E_{\lambda_j} - E_{\lambda_{j-1}}) \neq 0$, and likewise for some j satisfying $\lambda_j \geq \|Av\|$.*

Because $v \in \mathrm{ran}(\sigma(T))$, Lemma 4.5 implies $v = (E_{\|T\|} - E_0)v$. Thus since we suppose $E_{\lambda_1} = E_0$, $(E_{\lambda_1} - E_{\lambda_0})v = 0$, and we have the following decomposition of v:

$$v = (E_{\lambda_2} - E_{\lambda_1})v + \ldots + (E_{\lambda_n} - E_{\lambda_{n-1}})v.$$

Since this decomposition is orthogonal, the Pythagorean theorem applies: if we let c_j denote $\|(E_{\lambda_j} - E_{\lambda_{j-1}})v\|$ and take the squared norm of each side, we have

$$(1 =)\ \|v\|^2 = c_2^2 + \ldots + c_n^2.$$

Now by the definition of A (given with the definition of spectral resolutions above) and the nullity of $(E_{\lambda_1} - E_{\lambda_0})v$,

$$Av = \lambda_2(E_{\lambda_2} - E_{\lambda_1})v + \ldots + \lambda_n(E_{\lambda_n} - E_{\lambda_{n-1}})v.$$

Then by taking the squared norm of each side and invoking Pythagoras again, we have

$$\|Av\|^2 = c_2^2\lambda_2^2 + \ldots + c_n^2\lambda_n^2,$$

where the c_j^2 sum to 1. Thus $\|Av\|^2$ is a weighted average of the λ_j^2 with $2 \leq j \leq n$, so at least one $\lambda_j^2 \leq \|Av\|^2$ ($j \geq 2$) must have a nonzero coefficient c_j^2, and likewise for at least one $\lambda_j^2 \geq \|Av\|^2$. The lemma's claim follows. □

Lemma 4.8 *If* $P \not\leq Q$, *there exists* $P' \in \mathbb{P}^+$, $P' \leq P$, *satisfying* $\max(I_{P'}(Q)) < 1$.

We are concerned in this proof only with vectors in $\text{ran}(P)$, for which Lemma 4.3 ensures

$$\|Qv\| = \|Tv\|. \tag{1}$$

Lemma 4.4 ensures $\sigma(T) \leq P$. If $\sigma(T) \neq P$, we simply set $P' \equiv P - \sigma(T)$, so P' is orthogonal to T's support projection, implying that for all $v \in \text{ran}(P')$, $\|Qv\| = \|Tv\| = 0$; it follows that $\max(I_{P'}(Q)) = 0$, fulfilling the lemma's requirement.

Assume, then, that $\sigma(T) = P$. The idea now is to make P' an appropriate "slice" of the spectral resolution of T.

Since $P \not\leq Q$, there exist $\epsilon > 0$ and $v \in \text{ran}^{=1}(P)$ such that $\|Qv\| < 1 - \epsilon$. Fix some such v and ϵ; then by (1),

$$\|Tv\| < 1 - \epsilon. \tag{2}$$

By the norm-convergence of spectral approximations (see our definition of spectral resolution above) and by Lemma 4.6, there exist $n > 0$ and reals $\lambda_0 < \ldots < \lambda_n = \|T\|$, such that $\lambda_0 < 0, \lambda_1 = 0$, and when we define the approximation A to T by

$$A \equiv \sum_{j=1}^{n} \lambda_j(E_{\lambda_j} - E_{\lambda_{j-1}}), \tag{3}$$

we have $\|A - T\| < \epsilon/3$. This implies that for our unit vector v, $\|Av - Tv\| < \epsilon/3$, whence (by the triangle inequality)

$$\|Av\| \leq \|Tv\| + \epsilon/3, \tag{4}$$

which along with (2) implies

$$\|Av\| \leq 1 - 2\epsilon/3. \tag{5}$$

Now since $v \in \text{ran}^{=1}(\sigma(T))$ we can invoke Lemma 4.7 with v to obtain j such that $0 < \lambda_j \leq \|Av\|$ and $(E_{\lambda_j} - E_{\lambda_{j-1}}) \neq 0$. Choose the least such j and define

$$P' \equiv (E_{\lambda_j} - E_{\lambda_{j-1}}).$$

Since $0 < \lambda_j$ and we are supposing $\lambda_1 = 0$, we have $j \geq 2$ and $P' \leq (E_{\|T\|} - E_0)$. Thus since $P = \sigma(T)$ and $\sigma(T) = (E_{\|T\|} - E_0)$ (by Lemma 4.5), we have $P' \leq P$.

Fix an arbitrary $w \in \mathrm{ran}^{=1}(P')$. Since $P' = (E_{\lambda_j} - E_{\lambda_{j-1}})$, (3) entails $Aw = \lambda_j w$, whence by (5) and our choice of j,

$$\|Aw\| \;=\; \lambda_j \;\leq\; \|Av\| \;\leq\; 1 - 2\epsilon/3. \tag{6}$$

Now (1), (6), and the fact that $\|A - T\| < \epsilon/3$ give us

$$\|Qw\| \;=\; \|Tw\| \;\leq\; \|Aw\| + \epsilon/3 \;\leq\; 1 - (2\epsilon/3) + \epsilon/3 \;=\; 1 - \epsilon/3, \tag{7}$$

so $\|Qw\|^2 \leq (1 - \epsilon/3)^2$. Since $w \in \mathrm{ran}^{=1}(P')$ was arbitrary, $\max(I_{P'}(Q)) \leq (1 - \epsilon/3)^2$. □

Lemma 4.9 *For all $\epsilon > 0$ there exists $P' \leq P$, $P' \in \mathbb{P}^+$, such that $\min(I_{P'}(Q)) \geq \max(I_P(Q)) - \epsilon$.*

The proof here is essentially the same as Lemma 4.8's, but the demand that a specific bound ϵ be respected necessitates some extra arithmetic. To simplify this arithmetic later, let $\delta = \epsilon/(2\sqrt{\max(I_P(Q))})$. (Note we may assume $\max(I_P(Q)) > 0$ since otherwise the claim holds trivially.)

Just as in Lemma 4.8's proof, there exist $n > 0$ and reals $\lambda_0 < \ldots < \lambda_n = \|T\|$ determining a spectral approximation A to T such that $\lambda_0 < 0$, $\lambda_1 = 0$, and

$$\|A - T\| < \delta/3. \tag{8}$$

By definition, $\max(I_P(Q)) = \sup\{\|Qv\|^2 : v \in \mathrm{ran}^{=1}(P)\}$; this along with Lemma 4.3 yields

$$\max(I_P(Q)) = \sup\{\|Tv\|^2 : v \in \mathrm{ran}^{=1}(P)\}. \tag{9}$$

Note that Lemma 4.4 ensures $\sigma(T) \leq P$, and that $\|Tv\| = 0$ for all $v \in \mathrm{ran}(P - \sigma(T))$ (since such v are orthogonal to T's support projection); this implies

$$\max(I_P(Q)) = \sup\{\|Tv\|^2 : v \in \mathrm{ran}^{=1}(\sigma(T))\}. \tag{10}$$

There thus exists $v \in \mathrm{ran}^{=1}(\sigma(T))$ such that

$$\|Tv\| > \sqrt{\max(I_P(Q))} - \delta/3. \tag{11}$$

Fix such a v. This v, along with our λ_j's, meets the requirements to invoke Lemma 4.7, yielding j such that $(E_{\lambda_j} - E_{\lambda_{j-1}}) \neq 0$ and $\lambda_j \geq \|Av\|$. The latter along with (8) gives

$$\lambda_j > \|Tv\| - \delta/3. \tag{12}$$

Set $P' = (E_{\lambda_j} - E_{\lambda_{j-1}})$ for this j. Consider an arbitrary $w \in \mathrm{ran}^{=1}(P')$. By A's definition, $Aw = \lambda_j w$, so $\|Aw\| = \lambda_j$. This along with (8) and (12) give

$$\|Tw\| \;>\; \|Aw\| - \delta/3 \;=\; \lambda_j - \delta/3 \;>\; \|Tv\| - 2\delta/3. \tag{13}$$

This along with (11) entails

$$\|Tw\| > \sqrt{\max(I_P(Q))} - \delta. \tag{14}$$

Squaring both sides,

$$\|Tw\|^2 > \max(I_P(Q)) - 2\delta\sqrt{\max(I_P(Q))} + \delta^2 > \max(I_P(Q)) - 2\delta\sqrt{\max(I_P(Q))}. \tag{15}$$

By our propitious definition of δ, then, $\|Tw\|^2 > \max(I_P(Q)) - \epsilon$. Finally, since w is an arbitrary member of $\mathrm{ran}^{=1}(P')$ and all such members satisfy $\|Tw\|^2 = \|Qw\|^2$, we have

$$\min(I_{P'}(Q)) = \inf\{\|Qw\|^2 : w \in \mathrm{ran}^{=1}(P')\} \geq \max(I_P(Q)) - \epsilon. \quad \square \tag{16}$$

Theorem 4.10 *A generic filter $G \subseteq \mathbb{P}^+$ (with $\mathbb{P}$ the projection lattice of any $\mathcal{R}$ as stipulated at the outset) induces a generic probability measure μ_G on $\mathbb{P}$ with which G is interdefinable.*

We first prove that all $P \in \mathbb{P}^+$ satisfy the conditions of Lemma 3.3, and so they all force G to induce a generic probability measure. Fix arbitrary $P \in \mathbb{P}^+$, $\epsilon > 0$, and pairwise-orthogonal $\{Q_i : i < \alpha\} \subseteq \mathbb{P}$, with $0 < \alpha \leq \omega_0$; and let $Q_{<\alpha}$ denote their sum as in Lemma 3.3.

Case I. If α is finite, set $n = \alpha$; note that in this case, when we define $Q_{<n} \equiv \sum_{i<n} Q_i$ as in Lemma 3.3, we have $Q_{<n} = Q_{<\alpha}$, and so

$$\max(I_P(Q_{<n})) = \max(I_P(Q_{<\alpha})). \tag{17}$$

Case II. If $\alpha = \omega_0$, choose n as follows. From the definition of $I_P(Q)$, we have

$$\max(I_P(Q_{<\alpha})) = \sup\{\|Q_{<\alpha}v\|^2 : v \in \mathrm{ran}^{=1}(P)\}.$$

Choose some $w \in \mathrm{ran}^{=1}(P)$ such that

$$\|Q_{<\alpha}w\|^2 > \max(I_P(Q_{<\alpha})) - \epsilon/3.$$

Then let n be least such that

$$\|Q_{<n}w\|^2 > \|Q_{<\alpha}w\|^2 - \epsilon/3,$$

which is possible since $Q_{<n}$ converges to $Q_{<\alpha}$ in the strong operator topology as $n \to \infty$. The previous two inequalities, plus the definition of $I_P(Q)$ again, imply

$$\max(I_P(Q_{<n})) > \max(I_P(Q_{<\alpha})) - 2\epsilon/3. \tag{18}$$

In both cases I and II, we next invoke Lemma 4.9 to obtain $P' \leq P$ such that

$$\min(I_{P'}(Q_{<n})) \geq \max(I_P(Q_{<n})) - \epsilon/3.$$

From this and either (17) in case I or (18) in case II, we conclude that n and P' satisfy Lemma 3.3's requirement:

$$\min(I_{P'}(Q_{<n})) \geq \max(I_{P'}(Q_{<\alpha})) - \epsilon.$$

Thus G induces a generic probability measure μ_G.

It remains to show that G is definable from μ_G. We will do this by verifying

$$G = \{Q \in \mathbb{P}^+ : \mu_G(Q) = 1\}.$$

For all $Q \in \mathbb{P}^+$, $I_Q(Q) = \{1\}$; if $Q \in G$ then Lemma 3.2 entails $\mu_G(Q) = 1$. If $Q \notin G$ we must show $\mu_G(Q) \neq 1$. This will follow if the following union of sets is dense in $\mathbb{P}^+$:

$$\{P \in \mathbb{P}^+ : P \leq Q\} \cup \{P' \in \mathbb{P}^+ : \max(I_{P'}(Q)) < 1\}.$$

For if that holds, and we are supposing $Q \notin G$, then no member of the left-hand set can be in G (by the upward closure of filters), so some member of the right-hand set must be in G (by G's genericity), and this member forces $\mu_G(Q) < 1$. And indeed, Lemma 4.8 immediately entails that the above union is dense. □

5 Induced generic states

Since states on von Neumann algebras are of broader interest than probability measures on their projections, it is worth confirming that μ_G, the generic probability measure induced on $\mathbb{P}$ by G, extends to a unique state ω_G on $\mathcal{R}$ with which it is interdefinable.

The generalized Gleason theorem (see Theorem 12.3 of [5]) states that every probability measure on the projection lattice $\mathbb{P}$ of a von Neumann algebra $\mathcal{R}$ having no type I_2 summand extends to a unique state on $\mathcal{R}$. Alas, we cannot blithely apply this theorem to μ_G: we cannot invoke it in the ground model M, because μ_G does not exist there; and if we invoke it in $M[G]$, then our $\mathcal{R}$ may no longer be a von Neumann algebra, since it may not be closed in the strong operator topology (or even in the norm topology).

Instead of trying to modify the Gleason theorem to apply to our case, we will show how to explicitly define the state ω_G to which μ_G extends, using a modified version of Section 4's arguments to show that the vector states from vectors in the ranges of G's projections converge to ω_G. The main ideas are: (1) since states are complex-valued we need to constrain both the real and imaginary parts of their values; (2) since every operator has form $A+iB$ where A and B are self-adjoint it suffices (see Lemma 5.2) to show that the states' values converge on self-adjoint A; (3) since every self-adjoint A is (by the spectral theorem) approximated arbitrarily closely by finite linear sums of projections, we can use our previous result about the convergence of states' values on projections (Lemma 4.9) to obtain the same result for self-adjoint operators in general (Lemma 5.4).

Definitions. A *state* on $\mathcal{R}$ is a positive linear functional $\omega : \mathcal{R} \to \mathbb{C}$ satisfying $\omega(1) = 1$. The state on $\mathcal{R}$ induced by a unit vector $v \in H$ is $\omega_v : T \mapsto \langle Tv, v\rangle$. Note that μ_v, as we originally defined it, is ω_v's restriction to $\mathbb{P}$. $\mathrm{Re}(c)$ and $\mathrm{Im}(c)$ are the real and imaginary parts of $c \in \mathbb{C}$. For $P \in \mathbb{P}^+$ and $T \in \mathcal{R}$, define $I_P^{\mathrm{Re}}(T)$ to be the smallest closed interval of $\mathbb{R}$ containing the real parts of all values of $\omega_v(T)$ for $v \in \mathrm{ran}^{=1}(P)$, that is, the closed interval such that

$$\max(I_P^{\mathrm{Re}}(T)) = \sup\{\mathrm{Re}(\langle Tv, v\rangle) : v \in \mathrm{ran}^{=1}(P)\};$$

$$\min(I_P^{\mathrm{Re}}(T)) = \inf\{\mathrm{Re}(\langle Tv, v\rangle) : v \in \mathrm{ran}^{=1}(P)\}.$$

Similarly, let $I_P^{\mathrm{Im}}(T)$ be the smallest closed segment of the line $Re(x) = 0$ in the complex plane containing the imaginary parts of all values of $\omega_v(T)$ for $v \in \mathrm{ran}^{=1}(P)$.

Lemma 5.1 *For $T \in \mathcal{R}$, the intersection*

$$\bigcap_{P \in G} \overline{I_P^{\mathrm{Re}}(T)}$$

is a singleton $\{x_T\}$ (for some $x_T \in M[G]$) if and only if $(\forall \epsilon > 0)(\exists P \in G)(|I_P^{\mathrm{Re}}(T)| < \epsilon)$.

The proof is essentially the same as Lemma 3.2's. □

Definition. If for all $T \in \mathcal{R}$, the condition of Lemma 5.1 is satisfied both as written and with $I_P^{\mathrm{Im}}(T)$ in the place of $I_P^{\mathrm{Re}}(T)$, then the function $\omega_G : \mathcal{R} \to \mathbb{C}$ defined by

$$\omega_G(T) \equiv \text{ the sole member of } \bigcap_{P \in G} \overline{I_P^{\mathrm{Re}}(T)} \ + \text{ the sole member of } \bigcap_{P \in G} \overline{I_P^{\mathrm{Im}}(T)}$$

is well defined; if ω_G is a state on $\mathcal{R}$, we call it the *generic state induced by G on $\mathcal{R}$.*

Lemma 5.2 *If every self-adjoint $A \in \mathcal{R}$ meets the condition of Lemma 5.1, then G induces a generic state on $\mathcal{R}$.*

Suppose Lemma 5.1's condition holds for all self-adjoint A. We must first show that it holds, both as written and with $I_P^{\mathrm{Im}}(T)$ in the place of $I_P^{\mathrm{Re}}(T)$, for all $T \in \mathcal{R}$ including non-self-adjoint T; then ω_G as defined above will be well-defined. The key point is that any $T \in \mathcal{R}$ can be decomposed as $T = A + iB$, where A and B are both self-adjoint members of $\mathcal{R}$ (see [4], Section 2.4). For any state ω on $\mathcal{R}$, we have by linearity

$$\omega(T) = \omega(A) + \omega(iB) = \omega(A) + i\omega(B);$$

and then because states are real-valued on self-adjoint operators,

$$Re(\omega(T)) = \omega(A); \quad Im(\omega(T)) = i\omega(B).$$

This entails

$$I_P^{\mathrm{Re}}(T) = I_P^{\mathrm{Re}}(A); \text{ and}$$

$$I_P^{\mathrm{Im}}(T) = I_P^{\mathrm{Im}}(iB) = \{ix : x \in I_P^{\mathrm{Re}}(B)\}.$$

Thus since we are supposing Lemma 5.1's condition to hold for A and B, which are both self-adjoint, it holds for T in both its real and imaginary versions, and so G does induce a generic function ω_G on $\mathcal{R}$.

To show that ω_G is a state, observe that each ω_v in the set $\{\omega_v : v \in \mathrm{ran}^{=1}(P)\}$ used to define $I_P^{\mathrm{Re}}(T)$ and $I_P^{\mathrm{Im}}(T)$ satisfies finite additivity, positivity, and $\omega_v(1) = 1$. From this fact it is straightforward to prove that ω_G must satisfy these properties too. □

Lemma 5.3 *For all $Q \in \mathbb{P}$, $P \in \mathbb{P}^+$ and $\epsilon > 0$ there exists $P' \leq P$, $P' \in \mathbb{P}^+$, such that $|I_{P'}(Q))| < \epsilon$.*

This is an immediate corollary of Lemma 4.9. □

Lemma 5.4 *For all self-adjoint $A \in \mathcal{R}$, $\epsilon > 0$, and $P \in \mathbb{P}^+$, there exists $P' \in \mathbb{P}^+, P' \leq P$ such that $|I_{P'}^{\mathrm{Re}}(A)| < \epsilon$.*

Fix A, ϵ, and P. It follows easily from the definition of $I_{P'}^{\mathrm{Re}}(A)$ that all $P' \in \mathbb{P}^+$ satisfy

$$(\ (\forall v, w \in \mathrm{ran}^{=1}(P'))\ (|\langle Av, v\rangle - \langle Aw, w\rangle| < \epsilon/2)\ ) \ \Rightarrow\ |I_{P'}^{\mathrm{Re}}(A)| < \epsilon. \tag{19}$$

Suppose that $A' \in \mathcal{R}$ (an approximation of A) and unit vectors $v, w \in H$ satisfy

$$\|A' - A\| < \epsilon/6, \text{ and} \tag{20}$$

$$|\langle A'v, v\rangle - \langle A'w, w\rangle| < \epsilon/6. \tag{21}$$

By (20) we have

$$|\langle Av, v\rangle - \langle A'v, v\rangle| < \epsilon/6; \quad |\langle Aw, w\rangle - \langle A'w, w\rangle| < \epsilon/6. \tag{22}$$

From (21), (22), and a triangle inequality we conclude

$$|\langle Av, v\rangle - \langle Aw, w\rangle| < \epsilon/2. \tag{23}$$

In light of (19) and the fact that (20) and (21) imply (23), it now suffices to find $P' \leq P$ and $A' \in \mathcal{R}$ such that (20) holds for A' and (21) holds for all $v, w \in \mathrm{ran}^{=1}(P')$.

From the spectral approximation theorem (4.2), obtain $n > 1$ and $\lambda_0 < \lambda_1 < \ldots < \lambda_n$ determining a spectral approximation A' to A,

$$A' = \sum_{j=1}^{n} \lambda_j (E_{\lambda_j} - E_{\lambda_{j-1}}),$$

such that $\|A' - A\| < \epsilon/6$ and $\sum_{1\leq j\leq n}(E_{\lambda_j} - E_{\lambda_{j-1}}) = 1$. Then (20) holds and it remains to find P'. Note that for all $j, 1 \leq j \leq n$, $(E_{\lambda_j} - E_{\lambda_{j-1}})$ is a projection, and for all unit-norm $v \in H$, $\langle (E_{\lambda_j} - E_{\lambda_{j-1}})v, v\rangle = \|(E_{\lambda_j} - E_{\lambda_{j-1}})v\|^2$, and

$$\sum_{j=1}^{n} \|(E_{\lambda_j} - E_{\lambda_{j-1}})v\|^2 = \|v\|^2 = 1. \tag{24}$$

We now invoke Lemma 5.3 successively n times to obtain $P' \in \mathbb{P}^+, P' \leq P$ such that for all j,

$$|I_{P'}(E_{\lambda_j} - E_{\lambda_{j-1}})| < \epsilon/(6|\lambda_j| + 1).$$

The definition of $I_{P'}(Q)$ then implies that for any $v, w \in \text{ran}^{=1}(P')$,

$$|\|(E_{\lambda_j} - E_{\lambda_{j-1}})v\|^2 - \|(E_{\lambda_j} - E_{\lambda_{j-1}})w\|^2| \leq \epsilon/(6|\lambda_j| + 1). \tag{25}$$

From the mutual orthogonality of the projections $(E_{\lambda_j} - E_{\lambda_{j-1}})$ we have

$$\langle A'v, v\rangle = \sum_{j=1}^{n} \lambda_j \|(E_{\lambda_j} - E_{\lambda_{j-1}})v\|^2. \tag{26}$$

for all unit v. Finally, from (24), (25), and (26) it is straightforward to calculate that (21) holds for all $v, w \in \text{ran}^{=1}(P')$. □

Theorem 5.5 *A generic filter $G \subseteq \mathbb{P}^+$ (with $\mathbb{P}$ the projection lattice of any $\mathcal{R}$ as stipulated at the outset) induces a generic normal state ω_G on $\mathcal{R}$ with which it is interdefinable.*

By Lemma 5.4 and the genericity of G, every self-adjoint $A \in \mathcal{R}$ satisfies the condition of Lemma 5.1. Therefore by Lemma 5.2 a generic state ω_G is induced. For interdefinability, note that G is definable from μ_G (Theorem 4.10), which is definable from ω_G (being the latter's restriction to $\mathbb{P}$), which is defined from G.

The induced state ω_G is a *normal* state in the sense that it is countably additive on each set of mutually orthogonal projections whose join exists in $\mathbb{P}$; this is immediate from the fact that its restriction to $\mathbb{P}$ is μ_G, which is a probability measure on $\mathbb{P}$ as this term was defined in Section 2. (Note that the clause "whose join exists in $\mathbb{P}$" would be otiose here, as well as in our definition of probability measure, if we could assume $\mathcal{R}$ is a von Neumann algebra; but as mentioned at the outset of the present section, $\mathcal{R}$ may no longer be closed in the norm or strong operator topology in the generic extension $M[G]$). □

6 $\mathbb{P}$'s properties as a forcing notion

For clarity and simplicity we will restrict this section to cases where $\mathcal{R}$ is a *factor* (a von Neumann algebra with trivial center); type II and III factors are, in a well-known sense, the building blocks of noncommutative von Neumann algebras that have no minimal projections.

Fact 6.1 *When $\mathbb{P}$ is the projection lattice of a non-type-I factor $\mathcal{R}$, $\mathbb{P}^+$ is not c.c.c. (it is thus inequivalent to random-real forcing).*

By definition a poset meets the countable chain condition (or "is c.c.c.") if it has no uncountable antichain, and it is well known that $\mathbb{P}^+$ has an uncountable antichain when it is defined from a non-type-I factor $\mathcal{R}$. (Consider that such a factor always has a type I_2

subfactor, which is isomorphic to the algebra of all operators on a two-dimensional Hilbert space K; the projections along all the rays in K form an uncountable antichain.) $\Box$

Definitions. A poset $(Z, \leq)$ is a *real-number forcing poset* if any generic filter G on Z is interdefinable with a real number (or equivalently, if its boolean completion is countably-completely-generated). $\mathcal{R}$ is *almost-finite-dimensional* or A.F.D. if it is the closure in the strong operator topology of the union of some sequence $\mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \mathcal{N}_3 \subseteq \ldots$ of subalgebras where each $\mathcal{N}_n$ is a factor of type I_n.

Fact 6.2 *When $\mathbb{P}$ is the projection lattice of a non-type-I A.F.D. factor $\mathcal{R}$, $\mathbb{P}^+$ is a real-number forcing poset.*

Let G be any generic filter on such a $\mathbb{P}^+$. By Theorem 5.5, G is interdefinable with a normal state ω_G on $\mathcal{R}$; it thus suffices to show that a normal state on an A.F.D. algebra is interdefinable with a real number. Let $\mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \ldots$ be type I factors witnessing $\mathcal{R}$'s A.F.D. property, as defined just above. Since a type I_n factor is isomorphic to the algebra $M_n(\mathbb{C})$ of n-by-n matrices, it has a countable norm-dense subset (consider the set of matrices whose entries are rational in both their real and imaginary parts). Since the union of countably many countable sets is countable, there then exists a countable norm-dense subset X of $\bigcup_{i>0} \mathcal{N}_i$. Since X is countable and ω_G is a complex-valued function, ω_G's restriction to X can be encoded as a single real number with the usual kind of digit-interleaving method. Since states are norm-continuous and X is norm-dense in $\bigcup_{i>0} \mathcal{N}_i$, ω_G's values on the former determine all its values on the latter. Finally since ω_G is a *normal* state, it is continuous in the strong operator topology too; and so its values on $\bigcup_{i>0} \mathcal{N}_i$, which is s.o.t.-dense in $\mathcal{R}$, determine its values on all of $\mathcal{R}$. $\Box$

Question 6.3 *For what types of factors $\mathcal{R}$ does forcing with $\mathbb{P}^+$ collapse $2^{\aleph_0}$ (the cardinality of the continuum) to $\aleph_0$?*

Currently we do not even have a partial answer to this, i.e. for no (non-type-I) factor $\mathcal{R}$ do we know whether $\mathbb{P}^+$ will collapse $2^{\aleph_0}$. We will, however, register some basic observations that may help answer the question.

For cardinals $\kappa > \aleph_0$, let $(C_\kappa, \leq)$ denote the standard poset used for forcing the ground model's κ to become countable, namely, the set of finite sequences of ordinals $< \kappa$, ordered by reverse inclusion. Recall that when A, B are posets, a *dense embedding* $\phi : A \to B$ is an injective order-preserving map such that every $b \in B$ has some $a \in A$ satisfying $\phi(a) \leq b$.

Lemma 6.4 *Forcing with $\mathbb{P}^+$ is guaranteed (i.e. forced by all conditions) to collapse $2^{\aleph_0}$ to $\aleph_0$, if and only if $(C_{2^{\aleph_0}}, \leq)$ embeds densely into $\mathbb{P}^+$.*

This follows from Lemma 26.7 in [3], which proves that any forcing poset of cardinality $\kappa > \aleph_0$ that is guaranteed to collapse κ to $\aleph_0$ must densely embed a copy of $(C_\kappa, \leq)$. To apply this fact we must verify that $|\mathbb{P}^+| \leq 2^{\aleph_0}$; since $\mathbb{P}^+ \subseteq \mathcal{B}(H)$ it suffices to verify that $|\mathcal{B}(H)| \leq 2^{\aleph_0}$, using our global assumption that H is separable. Fix an orthonormal basis

$\{v_i\}$ for H and a countable dense subset $X \subseteq H$; let $\mathcal{F} \subseteq \mathcal{B}(H)$ be the set of operators that map $v_0, \ldots, v_n$ (for some finite n) to vectors in X and annihilate the other v_i; then note that $\mathcal{F}$ is countable and that every $T \in \mathcal{B}(H)$ is the strong-operator limit of some ω-sequence of $\mathcal{F}$-members. Since there are $2^{\aleph_0}$ possible ω-sequences drawn from a countably infinite set, $|\mathcal{B}(H)| \leq 2^{\aleph_0}$. $\square$

The following lemma may be useful towards proving that $(C_{2^{\aleph_0}}, \leq)$ does not embed densely into $\mathbb{P}^+$, if indeed there are any cases where it does not.

Lemma 6.5 *Fix a faithful normal state ω on $\mathcal{R}$, and suppose that for all maximal antichains $X \subseteq \mathbb{P}^+$ and all $P \in \mathbb{P}^+$, there exist $P' \in \mathbb{P}^+$, $P' \leq P$, with $\omega(P')$ arbitrarily close to $\omega(P)$, such that for all $Q \in X$, $Q \not\leq P'$. Then $\mathbb{P}^+$ does not admit a dense embedding of $(C_{2^{\aleph_0}}, \leq)$.*

Suppose the given condition holds and let $\phi : C_{2^{\aleph_0}} \to \mathbb{P}^+$ be an order-embedding. For $n > 0$ let X_n denote $\{\phi(c) : c \in C_{2^{\aleph_0}}, |c| = n\}$. Assume each X_n is a maximal antichain of $\mathbb{P}^+$ (otherwise the embedding ϕ is clearly not dense). Let $f : \mathbb{N} \to (0, 1)$ be a function such that $\prod_{i=1}^{n} f(i)$ converges to a positive number as $n \to \infty$. Set $P_0 = 1$ (the identity projection). At stage $n > 0$, invoke the given condition to obtain $P_n \in \mathbb{P}^+$, $P_n \leq P_{n-1}$, $\omega(P_n) > f(n)\omega(P_{n-1})$, and P_n is $\geq$ no X_n-member.

Since ω is a faithful normal state, it is strong-operator continuous, so the projection $Q \in \mathbb{P}$ to which the P_n converge in the strong operator topology will (by the constraint on f) satisfy $\omega(Q) > 0$; hence $Q \in \mathbb{P}^+$. Plainly no projection in ϕ's range is $\leq Q$, so ϕ is not a dense embedding. $\square$

On the other hand, suppose that for all maximal antichains $X \subseteq \mathbb{P}^+$ and all $Q \in \mathbb{P}^+$, there exists $P \in X$ such that either $P \leq Q$ or $P \geq Q$; then there is a dense embedding of C_κ into $\mathbb{P}^+$. (Proof: let X_0 be such an antichain; let X_{n+1} be a maximal antichain that "refines" X_n (i.e. every X_{n+1}-member is $<$ some X_n-member) and satisfies $\omega(P) < 1/(n+1)$ for all $P \in X_{n+1}$; then for any $Q \in \mathbb{P}^+$, consider X_n such that $1/(n+1) < \omega(Q)$ – some member of this X_n must be $\leq Q$ since none can be $\geq Q$.)

7 Appendix: the analogous construction using projections' ranges does not work

We remarked in the introduction that, assuming G is a generic filter of the projections in a commutative von Neumann algebra $\mathcal{R}$, if G's members were identified with their range spaces then they would *not* "shrink down to a generic limit point of the space" (as they do when each $P \in G$ is identified with the set of states on $\mathcal{R}$ induced by vectors in P's range). Here we will briefly substantiate this remark, for any $\mathcal{R}$ without minimal projections.

Recall that H denoted a Hilbert space in our ground model M. Let $\overline{H}$ be H's norm completion in $M[G]$, and for projections $P \in \mathbb{P}$, let $\overline{\mathrm{ran}(P)}$ be the closure of P's range in $\overline{H}$. By analogy with our definition of G's induced probability measure (in Section 3), let us define $l(G)$, G's *induced limit subspace*, by

$$l(G) \equiv \bigcap_{P \in G} \overline{\operatorname{ran}(P)}.$$

It is perhaps initially plausible that $l(G)$ could be a generic ray in $\overline{H}$, but in fact $l(G) = \{0\}$, as we now show.

Fact 7.1 *$l(G) = \{0\}$.*

Suppose some non-null vector v were in $l(G)$. We may assume $\|v\| = 1$ since $l(G)$ is clearly a linear subspace of $\overline{H}$. Since H is dense in $\overline{H}$, there exists some $w \in H$ and $\epsilon > 0$ such that $|\langle v, w\rangle| > \epsilon$. But let $P \in \mathbb{P}^+$ be arbitrary. Because $\mathbb{P}^+$ has no minimal projections, there exists a sequence $P > P_0 > P_1 > \ldots$ that is unbounded, and therefore converges to the null projection in the strong operator topology. This is the topology of pointwise convergence, so the sequence $P_0 w, P_1 w, P_2 w \ldots$ converges to the null vector. Thus there exists n such that $\|P_n w\| < \epsilon$. By the genericity of G, some $P \in G$ thus satisfies $\|Pw\| < \epsilon$, making it impossible for any unit vector $v \in \overline{\operatorname{ran}(P)}$ to satisfy $|\langle v, w\rangle| > \epsilon$. $\square$

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