Self-Constructing Continua

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Taking *continuum* to mean the set of all real numbers in a standard inner model of set theory, we axiomatize an intuitive notion of a set of distinct continua that "constructs itself," present a forcing construction that outputs candidates to satisfy our axioms, and identify the main requirement on a forcing poset for it to yield a model of the axioms when used in our construction. We do not yet know whether any forcing poset meets this requirement. On the negative side, we show that neither Cohen forcing nor random-real forcing can do so; on the positive side, we briefly discuss two types of forcing notions that might.

1 Self-Construction Axioms

By the *constructive closure* $\mathbb{R}(X)$ of a set X of real numbers we mean, informally, all the real numbers that must exist given that X does; formally, we define $\mathbb{R}(X)$ for any set X as the set of all reals in $L(X)$, the constructible hierarchy seeded with X's transitive closure and then built up with definable operations using Gödel's well-known procedure. For background on $L(X)$—which underlies everything that follows—see the appendix, Section 11.

By a *continuum* we mean a set X of real numbers (henceforth construed as subsets of ω) that is constructively closed, meaning $\mathbb{R}(X) = X$. Note that $L(X)$ is the smallest standard inner ZF model having X as a member (see Lemma 3.2) and it follows that we could define a continuum equivalently as the set of all real numbers in some standard inner ZF model.

$L(\emptyset)$, usually just denoted L, is the constructible universe of sets. Its continuum $\mathbb{R}(\emptyset)$ is the least continuum. Gödel famously proved it consistent with ZF that this is the *only* continuum.

It is also consistent with ZF that we can arrange distinct continua so that each is the constructive closure of the smaller ones' union, as in Example 2.1 below. Such an arrangement suggests the real numbers being constructed out of each other—growing over time, as it were—or, if the continua are not *linearly* ordered by inclusion, then growing over spacetime, as *it* were.

The goal of this paper is to determine whether this growth metaphor can be pushed further, whether a set of continua might grow "organically" without the need for anyone to do any preliminary "arranging." From this intuitive idea we will develop two axioms whose satisfaction would entitle a set of continua to be deemed *self-constructing*. However, as this development is necessarily somewhat discursive, we prefer to remove it to an appendix (Section 10) and to take the two axioms as our starting point. To state them we must first define "self-collection."

Definitions. Recall that a set $\mathcal{N}$ is *directed* (with respect to the inclusion ordering) if $X, Y \in \mathcal{N} \Rightarrow (\exists Z \in \mathcal{N})(X, Y \subseteq Z)$. We say that a set $\mathcal{N}$ of continua *self-collects into* X if the following hold:

(i) $X \notin \mathcal{N}$;

(ii) $\mathcal{N}$ is directed;

(iii) $\mathcal{N} \in L(X)$;

(iv) $(\neg\exists x \in X)(\bigcup\mathcal{N} \subseteq \mathbb{R}(x) \subset X)$;

(v) $X = \mathbb{R}(\bigcup\mathcal{N})$.

Axioms for a Self-Constructing Set $\mathcal{F}$ of Continua:

Self-Collection: $(\forall X \in L(\mathcal{F}))(X \in \mathcal{F} \iff$ some $\mathcal{N} \subseteq \mathcal{F}$ self-collects into $X)$.

Foundation: $(\forall X \in \mathcal{F})(\forall x \in X)(\exists Y \in \mathcal{F})$ $(x \in Y \subseteq X$ and $(\forall Z \in \mathcal{F})(Z \subset Y \Rightarrow x \notin Z))$.

As motivation for calling these the "self-construction" axioms, let us content ourselves for now to note that Self-Collection captures an idea of continua naturally forming a collection that requires vanishingly little work to constructively close; and that Foundation keeps real numbers from slipping in inexplicably or unaccountably during $\mathcal{F}$'s growth. Again, the full motivation is developed in Section 10.

It is easy to check that $\mathcal{F} = \{\mathbb{R}(\emptyset)\}$ is a self-constructing family. We turn now to the question of whether any non-trivial $\mathcal{F}$ can satisfy our two axioms.

2 Forcing Singly-Generated Continua

The technique of forcing is uniquely suited to produce candidates for satisfying the self-construction axioms. To illustrate the challenges involved in using it successfully to this end, we introduce two relatively simple examples from a Cohen-forcing extension, namely $\mathcal{C}_S$ and $\mathcal{C}_{\neg S}$.

Definition. A *singly-generated* continuum X is one that satisfies $X = \mathbb{R}(x)$ for some real number x (which is clearly never unique).

Example 2.1 *Let G be a generic filter for Cohen forcing (see [7], Chapter 15); let $\mathcal{C}_S$ be the set of singly-generated continua in $L[G]$; let $\mathcal{C}_{\neg S}$ be the set whose members are $\mathbb{R}(\emptyset)$ and all non-singly-generated continua in $L[G]$.*

Remark. Let us pause to stress that the relative constructibility used in our definitions has been the "parentheses" version ([7], Definition 13.24). A forcing extension $L[G]$ is defined differently, as the class of interpretations that G gives to specified forcing names in L. This generalizes to a "square-brackets" version of relative constructibility ([7], Definition 13.22) which guarantees for arbitrary sets Z that $L[Z]$ satisfies the Axiom of Choice (AC), but not that Z is actually a member of $L[Z]$; for our purposes, this would not have been to the point. The square-brackets version is, however, the standard one to use when discussing forcing extensions, as we will throughout this paper, so it is well worth pointing out that in most of the cases we will consider, $L(Z)$ and $L[Z]$ come to the same thing:

Fact (Lemma 11.5): $L(Z) = L[Z]$ for any set $Z \subseteq L$; this holds notably when Z is a single real number, or any filter on a constructible poset. □

Returning to our examples $\mathcal{C}_S$ and $\mathcal{C}_{\neg S}$, we cite the following facts, which are discussed in the appendix (Section 10):

• Both $\mathcal{C}_S$ and $\mathcal{C}_{\neg S}$ have the property that each of their members is the constructive closure of their smaller members' union.

• $\mathcal{C}_S$ satisfies our Foundation axiom but not Self-Collection.

• $\mathcal{C}_{\neg S}$ satisfies our Self-Collection axiom but not Foundation.

Given these two concrete examples, a natural strategy for forcing a self-constructing set of continua would be to choose one of them, and deduce what modifications would be neeeded to make it satisfy the axiom it does not already satisfy. We commit now to pursue this strategy with $\mathcal{C}_S$, first because we simply have no idea how to begin patching up $C_{\neg S}$'s failure to satisfy our Foundation axiom, and second, because the singly-generated continua in a forcing extension $L[G]$ have a useful simplifying feature. Namely, they are the continua of form $\mathbb{R}(G \cap D)$, where D is a countably-generated subalgebra of the boolean algebra on which G is a filter. (This is proved in Lemma 3.8; it is closely related to Lemma 15.43 of [7], the well-known "intermediate model theorem.") We will ensure by fiat that this simplifying feature is always available to us:

Stipulation 2.2 *Henceforth every self-constructing set $\mathcal{F}$ under consideration will belong to some forcing extension $L[G]$, and each $\mathcal{F}$-member will be singly generated.*

Our central question thus becomes how a set of continua under Stipulation 2.2 can satisfy the Self-Collection axiom. In particular we ask: If $L[G]$ is a generic extension of L by a filter G on boolean algebra B, what are the conditions on B, and on a B-name $\dot{\mathcal{N}}$ for a set of singly-generated continua, under which G's interpretation of $\dot{\mathcal{N}}$ will self-collect into $\mathbb{R}(G)$?

3 Main Definitions, Facts, Questions

We now turn to the definitions needed to answer the question just stated, and also to pose tighter versions of it, incorporating some simplifying suppositions about B and $\dot{\mathcal{N}}$. We will register these versions as our Main Question at the end of this section.

3.1 Definitions and Notation

$L(y)$ is the union of all levels $L_\alpha(y)$ of the (relative) constructive hierarchy seeded with $\{y\}$'s transitive closure (that is, $L_0(y) \equiv \mathsf{TrCl}(\{y\})$); see Section 11 for details and background.

When $x \in L(y)$ we say that y *constructs* x. If also $y \in L(x)$, we say x and y are *interconstructible.*

We will not restate the basic terminology of forcing. We assume familiarity with the boolean-algebraic approach to forcing used in T. Jech's standard text [7], as well as in J. Bell's text [2]. We side with [2] in preferring the symbols $\wedge$ and $\vee$ for boolean meet and join, whereas [7] generally uses $\cdot$ and $+$.

In what follows, B is always an atomless complete boolean algebra in L, the constructible universe. We should say that the structure $\langle B, 0, 1, \vee, \wedge, \neg\rangle$ is our boolean algebra and that B is just its underlying set, but we will conflate the two, as is customary. We also assume B is countably completely generated (see the definition just below).

An *ACSA* of B is an atomless complete subalgebra of B. Switching "atomless" and "complete" would yield the more attractive acronym "CASA", but "complete subalgebra of B" is a syntactic unit: it denotes not just any subalgebra C of B that is complete, but only those C wherein the supremum of any subset of C is the same whether calculated in C or in B, and likewise for infima. See [7], chapter 7, under "Complete and Regular Subalgebras."

If $X \subseteq B$, X *completely generates* the subalgebra $\overline{X}$ that is the smallest complete subalgebra of B that includes X.

$\mathsf{ACSAs}(B)$ denotes $\{C \in L : C$ is a countably-completely-generated ACSA of $B\}$. By *countably-completely-generated* we mean that $C = \overline{Q}$ for some countable $Q \subseteq C$. The restriction to such subalgebras is what will enforce Stipulation 2.2; see Lemma 3.8. As mentioned, we assume that B itself is countably-completely-generated.

C, D, E will always range over $\mathsf{ACSAs}(B)$.

B^+ means $B \setminus \{0\}$, that is, B without its least element.

G is an L-generic ultrafilter on B (equivalently, L-generic filter on B^+).

$C \leq_G D$ means $(G \cap D)$ constructs $(G \cap C)$, that is, $G \cap C \in L(G \cap D)$.

3.2 Main Facts About Relative Constructibility

Constructibility, relative and otherwise, is reviewed in more detail in the appendix, Section 11. The present section states some fundamental results, referring to that appendix for the proofs, and then derives from these results some basic facts about the interconstructibility of reals, continua, and generic filters.

Definition. A *standard ZF model* is one in which all elements are pure sets and "$\in$" is interpreted as the real membership relation among its elements. An *inner* ZF model is one that is transitive and has all the ordinals.

Definition. When y is any set, a *y-predicate* is a proposition $\Psi(x, c_1, \ldots, c_n)$ of the language of set theory, having one free variable x and finitely many constant symbols $c_1, \ldots, c_n$, considered together with fixed referents for the c_i chosen out of $L \cup \mathsf{TrCl}(\{y\})$. A y-predicate Ψ is *y-absolute* if

$$\{x : \Psi(x, c_1, \ldots, c_n)\}$$

evaluates to the same set in any standard inner ZF model having y as a member.

Lemma 3.1 *There is a proposition Ψ such that $\Psi(x, \alpha, y)$ is, for any ordinal α and set y, a y-absolute predicate satisfying $\{x : \Psi(x, \alpha, y)\} = L_\alpha(y)$.*

The proof is discussed in the appendix (Section 11). $\square$

Lemma 3.2 *$L(y)$ is the smallest standard inner ZF model having y as a member, in the sense that $L(y) \subseteq M$ for any other such model M.*

That $L(y)$ is a standard inner ZF model follows from a minor modification of the usual proof that L is such a model (see [7], Theorem 13.3, or [3], Theorem 1.2). The minimality claim for $L(y)$ follows from Lemma 3.1: If $L(y) \not\subseteq M$ then $L_\alpha(y) \notin M$ for some α, contradicting $L_\alpha(y)$'s definability by a y-absolute predicate. $\Box$.

Corollary 3.3 *The following are all easy consequences of Lemma 3.2:*

(a) If x constructs y (that is, $y \in L(x)$), then $L(y) \subseteq L(x)$.

(b) The relation "x constructs y" ($y \in L(x)$) is transitive.

(c) If x and y are interconstructible, then $L(x) = L(y)$ and $\mathbb{R}(x) = \mathbb{R}(y)$.

(d) The ordering $\leq_G$ on $\mathsf{ACSAs}(B)$ defined above (namely $C \leq_G D$ if and only if $G \cap C \in L(G \cap D)$) is guaranteed with boolean value 1 to be transitive; since it is also reflexive it is a preorder, so we may define $=_G$, $<_G$, etc. correlatively. $\Box$

It is a central fact about $L(y)$ that each of its members (not only each level $L_\alpha(y)$) is definable by a y-absolute predicate:

Lemma 3.4 *$z \in L(y)$ if and only if $z = \{x : \Psi(x, c_1, \ldots, c_n)\}$ for some y-absolute predicate $\Psi(x, c_1, \ldots, c_n)$.*

The proof is discussed in the appendix (Section 11). $\Box$

We now bring these central facts about relative constructibility to bear on our forcing construction (the generic filter G on the boolean algebra B).

Lemma 3.5 *For all (constructible) $X \subseteq B$, $G \cap \overline{X} \in L(G \cap X)$.*

By Lemma 3.4, $G \cap \overline{X} \in L(G \cap X)$ if and only if $G \cap \overline{X}$ is definable by a $(G \cap X)$-absolute predicate, which is what we will show.

$\overline{X}$, the boolean completion of X in B, was defined above as the smallest complete subalgebra of B containing X. The key to our proof is an alternative iterative definition of $\overline{X}$.

When $Y \subseteq B$, define $\phi(Y)$ to be Y's closure under arbitrary joins and negations in B, or more precisely:

$$\phi(Y) \equiv \{\bigvee S : S \subseteq Y \text{ and } S \in L\} \cup \{\neg z : z \in Y\}.$$

Now define a hierarchy by successive closures under ϕ:

$X_0 \equiv X;$

$X_{\alpha+1} \equiv \phi(X_\alpha)$;

$X_\alpha \equiv \bigcup_{\beta<\alpha} X_\beta$ when α is a limit ordinal.

It is easily checked that the union of all the X_α is the smallest complete subalgebra of B that includes X, as reckoned in L. In fact the growth of the X_α's must cease at some ordinal δ, lest $\overline{X}$'s cardinality exceed B's, so we may say that $\overline{X}$ is the union of the X_α's with α less than this δ. The function $\Xi : \alpha \mapsto X_\alpha$, with domain $\delta + 1$, is a member of L. (Lemmas 11.2 and 11.4 can be used to establish $\Xi \in L(\emptyset)$ rigorously.) Note that $\Xi(0) = X$ and $\Xi(\delta) = \overline{X}$.

Now for all $\alpha \leq \delta$ define $\Gamma(\alpha) \equiv G \cap \Xi(\alpha)$. Plainly we have $\Gamma(0) = G \cap X$ and $\Gamma(\delta) = G \cap \overline{X}$. Because G is a generic ultrafilter on B, we will have for all $\alpha < \delta$:

$$\Gamma(\alpha+1) = \{z \in \Xi(\alpha+1) : \neg z \in \Xi(\alpha) \setminus \Gamma(\alpha),$$
$$\text{or } (\exists S \in L, S \subseteq \Xi(\alpha))(z = \bigvee S \text{ and } S \cap \Gamma(\alpha) \neq \emptyset)\}.$$

(We have used here the fact that if $z = \bigvee S$ for some $S \in L, S \subseteq \Xi(\alpha)$, then those elements of B^+ that are $\leq$ some S-member form a subset of B^+ that is dense below z; hence by G's genericity, if $z \in G$, some S-member must be in $\Gamma(\alpha)$.)

Note now that $\Gamma(\alpha)$ is the union of previous $\Gamma(\beta)$'s when α is a limit ordinal; and that because the expresssion for $\Gamma(\alpha+1)$ above does not involve (either implicitly or explicitly) any unconstructible sets other than $\Gamma(\alpha)$, there is a proposition Ψ such that for all α, $\Psi(x, \Gamma(\alpha), \Xi, \alpha)$ is a $\Gamma(\alpha)$-absolute predicate that defines $\Gamma(\alpha+1)$. Lemmas 11.2 and 11.4 then ensure that $\Gamma(\delta)$ can be defined by a $\Gamma(0)$-absolute predicate, and since $\Gamma(\delta) = G \cap \overline{X}$ and $\Gamma(0) = G \cap X$, we are done. $\square$.

Lemma 3.6 *Every real number $x \in L[G] \setminus L$ is interconstructible with $G \cap D$ for some $D \in \mathsf{ACSAs}(B)$, and vice-versa.*

Fix $D \in \mathsf{ACSAs}(B)$ and (since the definition of $\mathsf{ACSAs}(B)$ requires D to be countably completely generated) fix $\{d_n : n \in \omega\} \subseteq D$ that completely generates D. By Lemma 3.5, $G \cap D \in L(G \cap \{d_n : n \in \omega\})$. If we define the real number x by $\{n : d_n \in G\}$, then x is clearly interconstructible with $G \cap \{d_n : n \in \omega\}$, so by Corollary 3.3 we have $G \cap D \in L(x)$. (And clearly $x \in L(G \cap D)$ since $x = \{n : d_n \in G \cap D\}$.)

Conversely, fix an unconstructible real number $x \subseteq \omega$. Let $\dot{x}$ be a B-name for x such that $1 \Vdash$ "$\dot{x} \subset \omega$ is unconstructible", and let D be the subalgebra of B completely generated by $\{\|n \in \dot{x}\| : n \in \omega\}$. D will be atomless because an atom of D would force for all n either $n \in \dot{x}$ or $n \notin \dot{x}$, thus forcing x to be a member of the ground model L, contradicting the guarantee that $\dot{x}$ names an unconstructible real. As in the previous paragraph, x will be interconstructible with $D \cap G$. $\square$

Lemma 3.7 *A real number x is interconstructible with its $\mathbb{R}(x)$.*

This is clear since $x \in \mathbb{R}(x) \subseteq L(\mathbb{R}(x))$, and $\mathbb{R}(x) \in L(x)$ by definition. $\square$

Lemma 3.8 *The singly-generated continua in a forcing extension $L[G]$ are precisely the sets of form $\mathbb{R}(G \cap D)$, where $D \in \mathsf{ACSAs}(B)$ or $D = \emptyset$.*

$\mathbb{R}(\emptyset)$ is obviously singly-generated since we are construing real numbers as subsets of ω and $\emptyset$ is such a subset.

Fix $D \in \mathsf{ACSAs}(B)$. By Lemma 3.6, $D \cap G$ is interconstructible with some $x \subseteq \omega$; and by Corollary 3.3, $\mathbb{R}(G \cap D)$ equals $\mathbb{R}(x)$, which is a singly-generated continuum. Conversely, fix an unconstructible singly-generated continuum $X \in L[G]$, and real number $x \subseteq \omega$ such that $X = \mathbb{R}(x)$. By the other direction of Lemma 3.6, x is interconstructible with some $G \cap D$, so again we will have $\mathbb{R}(x) = \mathbb{R}(G \cap D)$. $\square$

Corollary 3.9 *G is interconstructible with $\mathbb{R}(G)$ (since we assume B is countably completely generated).*

Our assumption that B is itself countably-completely-generated entails $B \in \mathsf{ACSAs}(B)$, so by Lemma 3.6 G is interconstructible with some real number x. By Lemma 3.7, x is interconstructible with $\mathbb{R}(x)$. By transitivity, G is interconstructible with $\mathbb{R}(x)$. By Corollary 3.3, $\mathbb{R}(G) = \mathbb{R}(\mathbb{R}(x)) = \mathbb{R}(x)$. The claim follows. $\square$

3.3 The Main Question

Plenty of forcing extensions $L[G]$ (including the Cohen-forcing extension in Example 2.1) have instances of $\mathcal{N}$ meeting requirements (i)-(iv) in the definition of self-collection into $\mathbb{R}(G)$. It is requirement (v), $\mathbb{R}(\bigcup \mathcal{N}) = \mathbb{R}(G)$, that is hard to satisfy; in order to focus on it we assume the following:

Stipulation 3.10 *$\mathcal{N}$ is a set of singly-generated continua meeting requirements (i)-(iv) in the definition of self-collecting into $\mathbb{R}(G)$, as well as Stipulation 2.2; furthermore, assume we have a name $\dot{\mathcal{N}}$ forced by $1 \in B$ to meet the foregoing.*

The question posed at the end of Section 2 now takes this form: Under what conditions will $\mathcal{N}$ meeting Stipulation 3.10 satisfy the final requirement for self-collection, $\mathbb{R}(\bigcup \mathcal{N}) = \mathbb{R}(G)$? This final requirement is, in light of the results of Section 3.2, equivalent to the following more perspicuous one:

Lemma 3.11 $\mathbb{R}(\bigcup\mathcal{N}) = \mathbb{R}(G) \iff G \in L(\bigcup\mathcal{N})$ *(under Stip. 3.10).*

Clause (iii) of Stipulation 3.10 is $\mathcal{N} \in L(\mathbb{R}(G))$. By Corollary 3.9 and Corollary 3.3 (c), $L(G) = L(\mathbb{R}(G))$. Thus $\mathcal{N} \in L(G)$, and since $L(G)$ satisfies the ZF axiom of union, $\bigcup\mathcal{N} \in L(G)$. If also $G \in L(\bigcup\mathcal{N})$, then G is interconstructible with $\bigcup\mathcal{N}$, and by Corollary 3.3 (c) we have $\mathbb{R}(\bigcup\mathcal{N}) = \mathbb{R}(G)$.

Conversely if $\mathbb{R}(\bigcup\mathcal{N}) = \mathbb{R}(G)$ then $\mathbb{R}(G) \in L(\bigcup\mathcal{N})$, whence by Corollary 3.9 and transitivity (Corollary 3.3 (b)) $G \in L(\bigcup\mathcal{N})$. □

This equivalence allows us to officially register our Main Question thus:

The Main Question ($\mathcal{N}$ Version, Forcing-Extension Perspective): *Assuming Stipulation 3.10, under what conditions does $G \in L(\bigcup\mathcal{N})$ hold (so that $\mathcal{N}$ fulfills the final requirement (v) for self-collection)?*

We will soon (Section 4) introduce a set Θ that is interconstructible with and more manageable than $\bigcup\mathcal{N}$, and we will find requirements on Θ equivalent to those of Stipulation 3.10 on $\mathcal{N}$. For the sake of keeping all versions of our Main Question in one place, we will register the second one now.

The Main Question (Θ Version, Forcing-Extension Perspective): *Assuming Stipulation 3.10, under what conditions does $G \in L(\Theta)$ hold?*

We will answer this version of the question in Theorem 6.8.

Note now that the question "Under what conditions does $\mathcal{N}$ self-collect into $\mathbb{R}(G)$?" is posed from the perspective of the forcing extension $L[G]$, and that we would ultimately like to answer a question posed from the perspective of the ground model L: "Under what conditions on B and on the B-name $\dot{\mathcal{N}}$ will $\dot{\mathcal{N}}$'s interpretation be *forced* to self-collect into $\mathbb{R}(G)$?"

The Main Question (Ground-Model Perspective): *How do we calculate $\|\dot{G} \in L(\bigcup\dot{\mathcal{N}})\|$ (under Stipulation 3.10) in terms of B and $\dot{\mathcal{N}}$?*

We will answer this version of the question in Theorem 7.7.

As to the question of whether these conditions can ever be met, the answer is yes; we will present in Section 9 an example of a set $\mathcal{N}$ of continua that self-collects. However, this example does not self-*construct* since for each $X \in \mathcal{N}$, no subset of X's predecessors (that is, of X's proper sub-continua in $\mathcal{N}$) self-collects into X. Prospects for obtaining models of the self-construction axioms are discussed in Section 9.

4 Equivalence Classes $\theta(X)$ Of Generic Filters

We now present the key to our analysis, which is a shift in perspective from reals and sets of reals, to sets of generic filters with which they are interconstructible. The reason for the shift is this: interconstructibility among reals and sets of reals does not reduce to any easily-verified isomorphism relation among them, but interconstructibility among generic filters in $L[G]$ does. To wit, if $F, F' \in L[G]$ are generic filters on (possibly distinct) subalgebras in $\mathsf{ACSAs}(B)$, F and F' are interconstructible if and only if they are "essentially isomorphic" in a way that we will made precise below with the term *copy*. (We should note that the basic idea here has long been understood; we introduce the new term "copy" only as a convenience.)

We already know (from Lemma 3.8) that a singly-generated continuum $Y \in L[G]$ is interconstructible with the filter $G \cap D$ for some subalgebra D of B. The D need not, however, be unique. Even worse (for the sake of simplicity), given an arbitrary set $\mathcal{C} \in L[G]$ of singly-generated continua, there need not exist a $\mathcal{C}$-absolute predicate that defines a choice function mapping each $Y \in \mathcal{C}$ to a single $G \cap D$ with which it is interconstructible. What *does* exist is a $\mathcal{C}$-absolute predicate for a function mapping each Y to an *equivalence class* of filters with which it is interconstructible, a "constructibility degree" of filters which we will call $\theta(Y)$:

$$\theta(Y) \equiv \{F \in L(Y) : F \text{ is a } L\text{-generic filter on an ACSA of } B \text{ and } Y \in L(F)\}.$$

Note that since Y is a singly-generated continuum, we could replace "$Y \in L(F)$" in this definition with "$Y = \mathbb{R}(F)$." Also note that Lemmas 3.6, 3.7, and 3.8 ensure that $\theta(Y)$ will never be empty (unless $Y = \mathbb{R}(\emptyset)$) since it has some $G \cap D$ as a member.

We now define Θ to be an interconstructible stand-in for $\bigcup \mathcal{N}$ built with these constructibility degrees of filters:

$$\Theta \equiv \{\theta(\mathbb{R}(x)) : x \in \bigcup \mathcal{N}\}.$$

Θ is indeed interconstructible with $\bigcup \mathcal{N}$, since in the other direction we have

$$\bigcup \mathcal{N} = \bigcup \{\mathbb{R}(M) : M \in \Theta\}.$$

It follows from the transitivity of constructibility (Corollary 3.3) that a given set is constructible from Θ if and only if it is constructible from $\bigcup \mathcal{N}$, and, in particular:

Lemma 4.1 *Under Stipulation 3.10, $\mathcal{N}$ satisfies $G \in L(\bigcup \mathcal{N})$ if and only if $G \in L(\Theta)$, where Θ is defined in terms of $\bigcup \mathcal{N}$ as above.* $\Box$

Since $\mathcal{N}$ meets Stipulation 3.10 with boolean (B) value 1, it is easy to verify that Θ as we have defined it correlatively to $\mathcal{N}$ will meet the following with boolean value 1:

Lemma 4.2 *Under Stipulation 3.10, the following assertions, corresponding to the first three requirements of the Self-Collection axiom, are forced by* $1 \in B$*:*

(i). $G \notin \bigcup \Theta$*;*

(ii). $(G \cap C, G \cap D \in \bigcup \Theta) \iff G \cap (\overline{C \cup D}) \in \bigcup \Theta$*;*

(iii). $\Theta \in L(G)$ *[trivial since* Θ *is* G*'s interpretation of a* B*-name].* $\Box$

We now define a function $\chi : \mathsf{ACSAs}(B) \to B$ to serve as a "boolean characteristic function" for $\bigcup \Theta$, or more precisely, to encode for each D whether $G \cap D$ is a member of $\bigcup \Theta$:

$$\chi(D) \equiv \|\dot{G} \cap D \in \bigcup \dot{\Theta}\|.$$

Lemma 4.3 *Under Stipulation 3.10 the following hold:*

(i). $\chi(B) = 0$*;*

(ii). $(\forall C, D)(\chi(C) \wedge \chi(D) = \chi(\overline{C \cup D}))$.

This is easy to verify since each claim about χ is a straightforward translation of the corresponding one on Θ in Lemma 4.2. $\Box$

Clause (ii) of Lemma 4.3 has the following useful consequence:

Lemma 4.4 *Clause (ii) on* Θ *(Lemma 4.3) entails*

$$(G \cap C \in \bigcup \Theta \text{ and } C \geq_G D) \Rightarrow G \cap D \in \bigcup \Theta;$$

similarly, clause (ii) on χ *(Lemma 4.3) entails*

$$(\forall C, D)(\chi(C) \wedge \|C \geq_G D\| \leq \chi(D)).\Box$$

Structure Of The Equivalence Classes $\theta(X)$

We now show that generic filters on $\mathsf{ACSAs}(B)$-members in $L[G]$ are interconstructible if and only if they are "essentially isomorphic" copies of each other; consequently, the constructibility degrees $\theta(Y)$ will be equivalence classes of copies.

Definitions:

$a, b \in B$ are *compatible* if $a \wedge b \neq 0$; similarly, a is compatible with $X \subseteq B$ if it is compatible with each X-member, and $X, Y \subseteq B$ are compatible if they are pairwise compatible.

The *upwards closure* of $X \subseteq B$ is $\{b \in B : (\exists x \in X)(x \leq b)\}$.

A *principal ideal* of $C \in \mathsf{ACSAs}(B)$ is $\{c \in C : c \leq e\}$ for some $e \in C^+$ (we put C^+ rather than C here because it will simplify matters to exclude the degenerate ideal $\{0\}$ by definition).

When $c \in C^+$, we use $C \restriction c$ to mean the principal ideal of C whose greatest member is c; and we write "$\max(I)$" for the greatest member of a principal ideal I.

Similarly, when ϕ is a function on C, we write $\phi \restriction c$ to mean ϕ's restriction to $C \restriction c$. (When context suggests that X is not a member of C, but rather a set likely to have a nonempty intersection with C, $\phi \restriction X$ will mean ϕ's restriction to $C \cap X$, as usual.)

Note that a principal ideal $C \restriction c$ can be considered a boolean algebra in its own right, having c as its greatest member, inheriting $\wedge$ and $\vee$ from C, and having a negation operation defined from C's negation $\neg$ as $e \mapsto c \wedge \neg e$.

When ϕ is a function, $\operatorname{dom} \phi$ and $\operatorname{ran} \phi$ denote its domain and range; and we adopt the common square-bracket notation $\phi[X]$ to mean $\operatorname{ran}(\phi \restriction X)$.

If $b \in B$ and $A \subseteq B$, then $b \wedge [A]$ means $\{b \wedge a : a \in A\}$.

$\mathsf{Aut}(B)$ is the set of automorphisms of B (bijections from B onto itself that preserve boolean structure) in the ground model L.

A *partial automorphism* of B is a bijection between principal ideals of (possibly distinct) ACSAs of B that preserves boolean structure (when the principal ideals are considered as boolean algebras in their own right, as described above).

$\mathsf{ParAut}(B)$ is the set of B's partial automorphisms in L.

Definition of "copy": When $C, D \in \mathsf{ACSAs}(B)$, and F is a filter on C, and $\phi \in \mathsf{ParAut}(B)$ is an isomorphism from some principal ideal of C not disjoint from F to some principal ideal of D, we say that ϕ *copies* F to the upward closure in D of $\phi[F \cap \operatorname{dom} \phi]$, and that this upward closure is a *copy* of F, *via* ϕ.

The importance of copies is established in the following lemma, which shows that for any real number x, its "constructibility degree" $\theta(\mathbb{R}(x))$ of filters is just an equivalence class of copies.

Lemma 4.5 *For each singly-generated unconstructible continuum $X \in L[G]$,*

$(\exists C)(G \cap C \in \theta(X))$, *and*

$(\forall C)(G \cap C \in \theta(X) \Rightarrow \theta(X) = \{F : F \text{ is a copy of } G \cap C\})$.

We will establish some preliminary lemmas in order to prove this.

Lemma 4.6 *If G_C, G_D are generic filters on (respectively) $C, D \in \mathsf{ACSAs}(B)$, then G_C constructs G_D (i.e. $G_D \in L(G_C)$) if and only if there exists $C' \in \mathsf{ACSAs}(B)$, $C' \subseteq C$, such that G_D is a copy of $G_C \cap C'$.*

The "if" direction is straightforward; we consider the "only if" direction. Suppose $G_D \in L(G_C)$. G_D is interconstructible with a real number r_D by Lemma 3.6. Therefore by transitivity of relative constructibility (Corollary 3.3 (b)) $r_D \in L(G_C)$. Again by Lemma 3.6, this time using our C for its B and our C_G for its G, we have r_D interconstructible with $L(G_C \cap C')$ for some $C' \in \mathsf{ACSAs}(C)$. Then by Corollary 3.3 (b) and (c), G_D is interconstructible with $G_C \cap C'$, and $L(G_D) = L(G_C \cap C')$. Note that $G_D, (G_C \cap C') \subseteq L$, so by Lemma 11.5, $L(G_D) = L[G_D]$ and $L(G_C \cap C') = L[G_C \cap C']$.

Since $G_D \in L[G_C \cap C']$ and $(G_C \cap C') \in L[D]$, there is a C'-name $\dot{G_D}$ whose interpretation by $G_C \cap C'$ is G_D, and a D-name $\dot{Q}$ whose interpretation by G_D is $G_C \cap C'$.

Let $\phi : D \to C'$ map each d to the boolean value of "d is a member of $\dot{G_D}$", that is, $\phi : d \mapsto \|\check{d} \in \dot{G_D}\|_{C'}$. Similarly let $\gamma : C' \to D$ be $\gamma : c \mapsto \|\check{c} \in \dot{Q}\|_D$.

It is straightforward to verify that ϕ and γ have the following properties:

(1) $d \leq d' \Rightarrow \phi(d) \leq \phi(d')$;

(2) $c \leq c' \Rightarrow \gamma(c) \leq \gamma(c')$;

(3) $d \in G_D \iff \phi(d) \in G_C \cap C'$;

(4) $c \in G_C \iff \gamma(c) \in G_D$.

From the last two properties we have

(5) $(\forall c \in C')(c \in G_C \iff \phi(\gamma(c)) \in G_C)$.

Now the genericity of G_C would guarantee a counterexample to (5) unless there existed $c \in G_C$ such that:

(6) $\phi(\gamma(c')) = c'$ for all $c' \in C' \restriction c$.

Similarly, because G_D is generic, we deduce the existence of $d \in G_D$ such that:

(7) $\gamma(\phi(d')) = d'$ for all $d' \in D \restriction d$.

Since $d \in G_D$ we have $\phi(d) \in G_C$ so that $\phi(d) \wedge c \in G_C \cap C'$.

We claim $\gamma \restriction (\phi(d) \wedge c)$ is a $\mathsf{ParAut}(B)$-member that copies $(G_C \cap C')$ to G_D. It is an isomorphism, that is, $c_1 \leq c_2 \iff \gamma(c_1) \leq \gamma(c_2)$, for (2) above establishes one direction, and in the other, if $c_1 \not\leq c_2$ but $\gamma(c_1) \leq \gamma(c_2)$, then (6) and (1) entail a contradiction. Finally, the range of $\gamma \restriction \phi(d) \wedge c$ is all of $D \restriction \gamma(\phi(d) \wedge c)$; if $d' < d$ were a counterexample to this, $\phi(d')$ would be in the domain of $\gamma \restriction \phi(d) \wedge c$ by (1), but then $\gamma(\phi(d')) \neq d'$, violating (7). $\square$

Lemma 4.7 *If in addition to the supposition of Lemma 4.6 we have $L(G_D) = L(G_C)$, we can require there that C' be C itself.*

At the outset of our proof we chose C' to be any ACSA of C such that $(G_C \cap C')$ is interconstructible with G_D, so knowing $L(G_D) = L(G_C)$ allows us to choose C itself. $\square$

Lemma 4.8 *A copy F' of a filter F on some $C \in \mathsf{ACSAs}(B)$ will be L-generic if and only if F is L-generic.*

Let ϕ copy F to F', and let $D \in \mathsf{ACSAs}(B)$ be such that $\operatorname{ran}\phi$ is a principal ideal of D, and F' is the upwards closure of $\phi[F \cap \operatorname{dom}\phi]$ in D.

We first show that F will be L-generic on C—that is, meet every constructible dense subset S of C^+—just if $F \cap \operatorname{dom}\phi$ is L-generic on $\operatorname{dom}\phi$. This is trivial if $\max(\operatorname{dom}\phi) = 1$, so assume otherwise. Note that $\max(\operatorname{dom}\phi) \in F$ by the "F not disjoint from dom ϕ" clause of the definition of "copy," and the fact that filters are upwards-closed.

Suppose F is L-generic on C and let S be any constructible dense subset of $\operatorname{dom}\phi$. Define $S' \equiv S \cup (C^+ \restriction \neg\max(\operatorname{dom}\phi))$. S' is dense in C^+, so there exists $s \in F \cap S'$; and this s must be in the S half of S' since otherwise, by the closure of F under $\wedge$, we would have $\max(\operatorname{dom}\phi) \wedge s = 0 \in F$, contradicting F's being a proper filter. Conversely, if $F \cap \operatorname{dom}\phi$ is L-generic on dom ϕ, and S is any constructible dense subset of C^+, define $S' \equiv \max(\operatorname{dom}\phi) \wedge [S]$; then S' is dense in $\operatorname{dom}\phi$, so there exists $s' \in F \cap S'$, which by definition of S' must be of form $s' = \max(\operatorname{dom}\phi) \wedge s$ for some $s \in S$, and this $s \in S$ must also be in F by the upward-closure of filters.

Likewise, F' is L-generic on D just if $F' \cap \operatorname{ran}\phi$ is L-generic on $\operatorname{ran}\phi$. The lemma then follows from the fact that ϕ is an isomorphism, which entails that a subset of $\operatorname{dom}\phi$ will be dense just if its image is dense in $\operatorname{ran}\phi$. □

Proof of Lemma 4.5:

"$\exists C$" clause: Just after defining $\theta(X)$ above, we noted that it always has a member of form $G \cap C$ (unless X is constructible, a case that Lemma 4.5's statement explicitly excludes).

"$\forall C$" clause: Letting C be arbitrary such that $G \cap C \in \theta(X)$, $G \cap C$ is of the form stated in the lemma because it is trivially a copy of itself; any other copy of $G \cap C$ is L-generic by Lemma 4.8 and is clearly interconstructible with $G \cap C$, so satisfies the definition of a $\theta(X)$-member. Finally, it follows from Lemma 4.6 and Lemma 4.7 that every $F \in \theta(X)$ is of this form. □

The following lemma gives more information about an important special case of Lemma 4.6, in which G_C and G_D are the restrictions to C and D of our generic G on B.

Lemma 4.9 $D \leq_G C \iff (\exists g \in G)(g \wedge [D] \subseteq g \wedge [C])$.

Recall that $D \leq_G C$ by definition means $G \cap D \in L(G \cap C)$, and that $L(G \cap C) = L[G \cap C]$ (Lemma 11.5, cited previously).

$\Rightarrow$: There is a C-name (a B-name involving only elements of C) for $G \cap D$; call it $\dot{\delta}$. Under interpretation by G, $\dot{\delta}$ names the same set as the canonical B-name for $G \cap D$, so we must have

$$\|\dot{\delta} = \dot{G} \cap \check{D}\|_B \in G.$$

Let g denote $\|\dot{\delta} = \dot{G} \cap \check{D}\|_B$. We verify that g meets the lemma's requirement. For all $d \in D$ define $\phi(d)$ by

$$\phi(d) \equiv \bigvee\{c \in C : c \Vdash \check{d} \in \dot{\delta}\}.$$

Consider the "nice" C-name $\dot{\eta} = \{(\check{d}, \phi(d)) : d \in D\}$. It is easy to verify that $\|\dot{\eta} = \dot{G} \cap \check{D}\|_B = g$. Suppose there is some $d \in D$ with no corresponding $c \in C$ such that $d \wedge g = c \wedge g$. Then $(g \wedge ((d \wedge \neg\phi(d)) \vee (\phi(d) \wedge \neg d)))$ is nonzero, and is an element of B below g that forces $\dot{\eta} \neq \dot{G} \cap \check{D}$, impossible.

$\Leftarrow$: $\dot{\eta}$ as defined above is a C-name whose interpretation by $G \cap C$ is $G \cap D$, in light of $\|\dot{\eta} = \dot{G} \cap \check{D}\|_B = g \in G$. □

We can express the meaning of Lemma 4.9 by saying: there is some element g of information in the filter G, such that any *additional* information about G's extension that might be supplied by $G \cap D$, could just as well be supplied by $G \cap C$. This suggests the following terminology.

Definition: Call $b \in B^+$ an *information supplement for C's generic to construct D's generic* if $b \wedge [D] \subseteq b \wedge [C]$. Let $IS(C, D)$ denote the set of all such information supplements; it is clearly a downward-closed subset of B^+. With this definition we can restate Lemma 4.9 thus:

Corollary 4.10 $\|C \geq_G D\| = \bigvee IS(C, D)$. $\Box$

5 Self-Collection Condition In Terms Of Definability

We are now ready to begin answering the Main Question (in its "Θ version") stated at the end of Section 3.3, from the perspective of the generic extension $L[G]$. The question is under what conditions Θ obeying Stipulation 3.10 (see Lemma 4.2) constructs G, meaning $G \in L(\Theta)$.

Our starting point is an answer we have already found, in terms of definability: By Lemma 3.4, $G \in L(\Theta)$ if and only if G is definable by a Θ-absolute predicate. This answer is not very helpful by itself, since there is no obvious way to check whether this condition holds. We will now begin the work of translating it into a more easily verifiable condition, work which will culminate in Theorem 6.8. The first step is to obtain a somewhat tighter definability condition.

Lemma 5.1 *Under Stipulation 3.10, $G \in L(\Theta)$ if and only if G is definable by some Θ-absolute predicate of form*

$$\Psi(x, \Theta, G \cap C, c'_1, \ldots c'_n),$$

for some finite set of constants $c'_1, \ldots c'_n$ (denoting sets) in L and some (constant denoting) $G \cap C \in \bigcup \Theta$.

The "only if" direction is what needs to be proved. Assume $G \in L(\Theta)$ and (invoking Lemma 3.4) that G is definable by a Θ-absolute predicate as

$$G = \{x : \Delta(x, c_1, \ldots c_n)\}.$$

We wish to show that there exists a Θ-absolute predicate of the more narrowly-defined form demanded in the lemma's statement, that defines the same set (G).

The idea here is as follows. Δ's given constants $c_1, \ldots, c_n$ must refer, by the definition of Θ-absolute predicate, to sets that belong to L and/or to $\mathsf{TrCl}(\{\Theta\})$. We wish to find a *single* constant referring to some $(G \cap C) \in \mathsf{TrCl}(\{\Theta\})$, such that each c_i can be replaced in Δ by a subformula that defines c_i's referent using only Θ, this $G \cap C$, and finitely many constants *required to be members of* L, all of which will be parameters for our Ψ. What allows us to do this is the fact (established by Lemma 4.5) that each given c_i that is not already an L-member must be interconstructible with $(G \cap C_i)$ for some $C_i \in \mathsf{ACSAs}(B)$ such that $(G \cap C_i) \in \bigcup(\Theta)$. Clause (ii) of Lemma 4.2 on Θ (which translates the directedness requirement on $\mathcal{N}$) then guarantees the existence of a single $C \in \mathsf{ACSAs}(B)$ that includes each such C_i, and satisfies $(G \cap C) \in \bigcup \Theta$. Each of the $(G \cap C_i)$'s is then definable as the intersection with C_i of this one $(G \cap C)$. And each C_i is an L-member and so can be appended as another constant for our Ψ.

We formalize this argument in the appendix, Section 11. □

Necessary Conditions For Self-Collection

An immediate consequence of Lemma 5.1 is the following necessary condition for self-collection:

Lemma 5.2 *If a proposition Ψ, an ACSA C, and a finite sequence $\vec{c}$ of constants in L jointly witness $G \in L(\Theta)$ as in Lemma 5.1, there cannot exist in any ZF model an L-generic filter G' on B, $G' \neq G$, such that $G' \cap C = G \cap C$ and G' gives the same interpretation as G to the B-name $\dot{\Theta}$.*

For if such a G' existed, and were used to interpret B-names to yield a generic extension of L, the expression

$$\{x \in B : \Psi(x, \Theta, G \cap C, \vec{c})\}$$

would evaluate there to $G' \neq G$, contradicting the Θ-absoluteness of the predicate. □

The necessary condition just stated entails another one, more explicitly connected to the structure of B and χ, which we will phrase in terms of "(C, χ)-respecting automorphisms" of B:

Definition. If $\phi \in \mathsf{Aut}(B)$, we call ϕ (C, χ)*-respecting* if it leaves every C-member fixed and for all $D \in \mathsf{ACSAs}(B)$, $\phi(\chi(D)) = \chi(\phi[D])$.

Fact 5.3 *If $G \in L(\Theta)$, and $C \in \mathsf{ACSAs}(B)$ with $\chi(C) \in C$ witnesses Lemma 5.1, then there exists no nontrivial (that is, non-identity) (C, χ)-respecting $\phi \in \mathsf{Aut}(B)$.*

Supposing the forbidden C and ϕ did exist, consider the filter $G' = \phi[G]$. It too is L-generic (Lemma 4.8). Because ϕ respects C, we have $G \cap C = G' \cap C$, and because ϕ exists in the ground model and respects χ, $\dot{\Theta}$'s interpretation is the same under G' as under G, as is easily verified. But then G' violates Lemma 5.2. □

Remark. We do not know at present whether the non-existence of (C, χ)-respecting automorphisms in the ground model, as in Fact 5.3, is *sufficient* for self-collection. Matters would be greatly simplified if it were. In our proof of Theorem 6.8, though, we will see that it may be possible to cobble together some of B's *partial* automorphisms—in a model larger than $L[G]$—to obtain a filter G' that satisfies Lemma 5.2, and so witnesses the failure of self-collection, yet is *not* isomorphic to G via any ground-model isomorphism. But perhaps the possibility of obtaining a G' this way implies that a (C, χ)-respecting automorphism already existed in the ground model.

In any case, the non-existence of (C, χ)-respecting automorphisms in the ground model L would not supply any obvious formula Ψ for defining G from Θ and $G \cap C$ (plus finitely many constants in L). In the next sections we will find a necessary and sufficient condition for self-collection that does provide such a Ψ.

6 Self-Collection Condition in Terms of Filters

The concept we use for turning Section 5's self-collection condition into a practically verifiable one is "generic refinability." We will define this concept in parallel for both generic filters on ACSA's of B and $\mathsf{ParAut}(B)$-members. Having done so, we will derive a self-collection condition in terms of generically-refinable filters that can be checked from the generic-extension perspective; then we will translate it into a condition in terms of generically-refinable $\mathsf{ParAut}(B)$-members, which can be checked in the ground model perspective. (Note that this second step is a standard procedure in the theory of forcing, namely the deduction of how the structure of one's forcing poset controls the structure of sets in the generic extension.)

Before defining generic refinability we introduce one-element refinements.

One-element refinements

It will be important for us to be able to expand an ACSA by adding a single B-member to it and taking its completion. We use the following notation for this:

When $C \in \mathsf{ACSAs}(B)$ and $b \in B$,

$$C^{+b} \equiv \{(c_1 \wedge b) \vee (c_2 \wedge \neg b) : c_1, c_2 \in C\}.$$

Lemma 6.1 *If $C \in \mathsf{ACSAs}(B)$ and $b \in B$, then $C^{+b} \in \mathsf{ACSAs}(B)$.*

This follows from routine boolean-algebraic manipulations, which we relegate to an appendix (Section 11). □

Refining. If F, F' are filters, we say F' *refines* F if $F \subseteq F'$. If $\phi, \phi' \in \mathsf{ParAut}(B)$, we say ϕ' *refines* ϕ if:

$\max(\mathrm{dom}\,\phi') \leq \max(\mathrm{dom}\,\phi)$, and $\max(\mathrm{ran}\,\phi') \leq \max(\mathrm{ran}\,\phi)$; and

$(\forall e \in \mathrm{dom}\,\phi)(\phi'(e \wedge \max(\mathrm{dom}\,\phi')) = \phi(e) \wedge \max(\mathrm{ran}\,\phi'))$.

Note the second clause implies $\max(\mathrm{dom}\,\phi') \wedge [\mathrm{dom}\,\phi] \subseteq \mathrm{dom}\,\phi'$.

Definition. When $b \in B$ and C is an ACSA of B — or, more generally, a subset of B that is closed under $\bigwedge$ (arbitrary meets) — we define

$$b \uparrow C \equiv \bigwedge \{c \in C : b \leq c\}.$$

Given $\phi \in \mathsf{ParAut}(B)$ and some $b \in B^+$, $b \leq \max(\mathrm{dom}\,\phi)$, a *one-element domain refinement* of ϕ by b is a refinement ϕ^{+b} of ϕ satisfying $\mathrm{dom}\,\phi^{+b} = b \wedge [\mathrm{dom}\,\phi]$ and $\max(\mathrm{ran}\,\phi^{+b}) = \phi(b \uparrow \mathrm{dom}\,\phi)$. We will now see that ϕ^{+b} is unique.

Lemma 6.2 *If $C \in \mathsf{ACSAs(B)}$ and $b \in B^+$, the function $\psi : a \mapsto a \wedge b$ with $\mathrm{dom}\,\psi = C \upharpoonright (b \uparrow C)$ is an isomorphism such that $\psi \in \mathsf{ParAut}(B)$, and $\psi(a) \uparrow C = a$ for all $a \in \mathrm{dom}\,\psi$.*

If we can show

$$(*) \quad (a \wedge b) \uparrow C = a$$

for all $a \in \mathrm{dom}\,\psi$, then the rest of the lemma's claims will follow easily. Let $a \in \mathrm{dom}\,\psi$ be arbitrary, so that $a \in C$ and $a \leq b \uparrow C$. Plainly we have $\leq$ for (*); we must show that this inequality cannot be strict. Suppose

$$(a \wedge b) \uparrow C = a' < a.$$

It would follow that $b \wedge a = b \wedge a'$. Then routine boolean algebra gives us the following chain of equalities:

$$\begin{aligned} b &= (b \wedge a) \vee (b \wedge \neg a) \\ &= (b \wedge a') \vee (b \wedge \neg a) \\ &= b \wedge (a' \vee \neg a). \end{aligned}$$

Thus $b \leq (a' \vee \neg a)$. But then also $b \leq (a' \vee \neg a) \wedge (b \uparrow C)$; the right side of that inequality is strictly $< (b \uparrow C)$, contradicting the definition of $(b \uparrow C)$ as the least C-member that is $\geq b$. □

Lemma 6.3 *Given $\phi \in \mathsf{ParAut}(B)$ and some $b \in B^+$, $b \leq \max(\operatorname{dom}\phi)$, the one-element domain refinement ϕ^{+b} exists and is unique.*

Let C be any ACSA such that $\operatorname{dom}\phi \subseteq C$, and note that $b \wedge [\operatorname{dom}\phi]$ $(= b \wedge [C])$ is a principal ideal of C^{+b} defined above.

By Lemma 6.2, the map $\psi : a \mapsto a \wedge b$ is an isomorphism mapping $C \restriction (b \uparrow C)$ onto $b \wedge [C]$, the latter being a principal ideal of C^{+b}. Thus we may define ϕ^{+b} by composition:

$$\phi^{+b}(d) \equiv \phi(\psi^{-1}(d)),$$

with the domain of ϕ^{+b} set to $b \wedge [C]$. It is straightforward to check that ϕ^{+b} meets the definition of a refinement of ϕ, and that it is the unique such refinement satisfying the definition of a one-element refinement of ϕ by b. □

In light of Lemma 6.3 we may use ϕ^{+b} to denote *the* one-element domain refinement of ϕ by b.

Similarly, for $b \leq \max(\operatorname{ran}\phi)$, we define the *one-element range refinement* ϕ_{+b} of ϕ to be the refinement of ϕ satisfying $\operatorname{ran}\phi_{+b} = b \wedge [\operatorname{ran}\phi]$ and $\max(\operatorname{dom}\phi_{+b}) = \phi^{-1}(b \uparrow \operatorname{ran}\phi)$.

Lemma 6.4 *For all $C \in \mathsf{ACSAs}(B)$ and $b \in B^+$, $\|C =_G C^{+b}\| = 1$, so $\chi(C) = \chi(C^{+b})$.*

Suppose $b \in G$. By Lemma 6.2, the map $a \mapsto a \wedge b$ is a member of $\mathsf{ParAut}(B)$ mapping $C \restriction (b \uparrow C)$ onto $b \wedge [C]$, the latter being a principal ideal of C^{+b}. Since $b \in G$, neither the domain of this map nor its range is disjoint from G, so this map copies $G \cap C$ to $G \cap C^{+b}$, and its inverse maps $G \cap C^{+b}$ to $G \cap C$. Thus by Lemma 4.6, $G \cap C^{+b}$ and $G \cap C$ are interconstructible; so by definition $C =_G C^{+b}$.

If on the other hand $\neg b \in G$, the same argument goes through with the map $a \mapsto a \wedge \neg b$. Thus both b and $\neg b$ force $C =_G C^{+b}$; thus 1 forces this too. $\chi(C) = \chi(C^{+b})$ then follows from Lemma 4.4. □

Generic Refinability In Accordance With χ

Remark. The definitions of Γ_α, Γ, Φ_α, and Φ below are cumbersome, because they are recursive; it is a question we have not been able to settle whether there are simpler definitions that would be equivalent. We will see that if $\Phi_1 = \Phi_0$, then $\Phi = \Phi_0$ (Lemma 8.4), but we have found no comparable simplification for other cases.

Accordance with χ. If F is an ultrafilter on $C \in \mathsf{ACSAs}(B)$, we say it *accords with* χ if $\chi(C)$ is in the upwards closure of F in B. For $\phi \in \mathsf{ParAut}(B)$, we say ϕ *accords with* χ if its domain and range are principal ideals of some C and D (respectively) such that $\max(\operatorname{dom}\phi) \leq \chi(C)$ and $\max(\operatorname{ran}\phi) \leq \chi(D)$. Note C and D may not be uniquely determined (since two distinct subalgebras of B can have a principal ideal in common), but the definition of χ ensures that if this requirement holds for one choice of C, D, it holds for all choices. This is because whenever $\operatorname{dom}\phi$ is a principal ideal of both C and C', we have $\max(\operatorname{dom}\phi) \Vdash C =_G C'$ (by Lemma 4.9), so by Lemma 4.4 for χ,

$$\max(\text{dom } \phi) \wedge \chi(C) = \max(\text{dom } \phi) \wedge \chi(C'),$$

and similarly for the range of ϕ.

Generic refinability. $\phi \in \mathsf{ParAut}(B)$ is *generically refinable within a class* $W \subseteq \mathsf{ParAut}(B)$ if:

$(\forall C, D \in \mathsf{ACSAs}(B))(\forall S \subseteq B^+, S \in L$, with S dense in $B^+)$

$(\chi(C)$ compatible with $\max(\operatorname{dom}\phi) \Rightarrow (\exists \phi' \in W, s \in S)$

$(\phi'$ refines ϕ, and

$\max(\operatorname{ran}\phi') \leq s$, and

$\max(\operatorname{dom}\phi') \wedge [C] \subseteq \operatorname{dom}\phi'$, and

$(\max(\operatorname{ran}\phi') \leq \neg\chi(D)$, or $\max(\operatorname{ran}\phi') \wedge [D] \subseteq \operatorname{ran}\phi')))$.

Similarly, a filter $F \in \bigcup\Theta$ is *generically refinable in* (W, Θ), where $W \subseteq \bigcup\Theta$ if:

$(\forall D \in \mathsf{ACSAs}(B))(\forall M \in \Theta, S \subseteq B^+, S \in L$, with S dense in $B^+)$

$(\exists N \in \Theta, F' \in N \cap W, s \in S \cap F')$

$(M \in L(N)$ and F' refines F and

$(\neg\chi(D) \in F'$, or $F' \cap D$ is an L-generic filter on $D))$.

The set $\Gamma(\Theta)$ of fully refinable filters; the set Φ of fully refinable partial automorphisms

In the context of a particular χ, we use the above definitions to define a $\Gamma_\alpha(\Theta)$ hierarchy somewhat along the lines of $L_\alpha(\Theta)$, except here the sequences of levels shrink rather than grow:

$\Gamma_0(\Theta) \equiv \{F \in \bigcup \Theta : F \text{ accords with } \chi\}$;

$\Gamma_{\alpha+1}(\Theta) \equiv \{F \in \Gamma_\alpha : F \text{ is generically refinable in } (\Gamma_\alpha, \Theta)\}$;

$\Gamma_\alpha(\Theta) \equiv \bigcap_{\beta<\alpha} \Gamma_\beta(\Theta)$ for limit α;

$\Gamma(\Theta) \equiv \bigcap_{\alpha \in Ord} \Gamma_\alpha(\Theta)$.

We call $\Gamma(\Theta)$ the set of "fully generically-refinable" filters (in the context of a given χ and Θ); similarly, we define the set Φ of fully generically-refinable partial automorphisms:

$\Phi_0 \equiv \{\phi \in \mathsf{ParAut}(B) : \phi \text{ accords with } \chi\}$;

$\Phi_{\alpha+1} \equiv \{\phi \in \Phi_\alpha : \phi \text{ is generically refinable within } \Phi_\alpha\}$;

$\Phi_\alpha \equiv \bigcap_{\beta<\alpha} \Phi_\beta$ for limit α;

$\Phi \equiv \bigcap_{\alpha \in Ord} \Phi_\alpha$.

Clearly we have:

Lemma 6.5 *Each $\Gamma(\Theta)$-member is generically refinable in $(\Gamma(\Theta), \Theta)$; and each Φ-member is generically refinable within Φ.* $\square$

Lemma 6.6 *For all α, $\Phi_\alpha \in L$ and $\Gamma_\alpha(\Theta) \in L(\Theta)$; moreover, $\Phi \in L$ and $\Gamma(\Theta) \in L(\Theta)$.*

The proof is an application of our template for showing absoluteness of hierarchies, Lemmas 11.2 and 11.4. We show how to apply it to the $\Gamma(\Theta)$ hierarchy; the proof for Φ is similar. The template takes two predicates, $\Omega(\cdots)$ and $\Psi(\cdots)$, as its "inputs". For the first, let $\Omega(x', \Theta)$ be "$x' \in \bigcup \Theta$ and x' accords with χ"; and for the second, let $\Psi(x'', (Y, \Theta, \beta))$ be "$x'' \in Y \cap \bigcup \Theta$ and x'' is generically-refinable in $((Y \cap \bigcup \Theta), \Theta)$."

We affirm first that our $\Omega(x', \Theta)$ is a Θ-absolute predicate, since the union operation is absolute and χ is a member of L. We affirm secondly that our $\Psi(x'', (Y, \Theta, \beta))$ is (Y, Θ, β)-absolute: this is because the definition of generic refinability for filters quantifies only over Θ, Y, and constructible subsets of B^+.

With the claim about the $\Gamma_\alpha(\Theta)$ proved, the claim about $\Gamma(\Theta)$ follows from the observation that the levels $\Gamma_\alpha(\Theta)$ must stop shrinking when α reaches some ordinal δ, so that $\Gamma(\Theta) = \Gamma_\delta(\Theta)$. Similarly for Φ. $\square$

The next lemma establishes that $\Gamma(\Theta)$ is not empty, and in particular has at least one filter of form $G \cap C$ from each Θ-member.

Lemma 6.7 *$(G \cap C) \in M \in \Theta \Rightarrow (G \cap C^{+\chi(C)}) \in M \cap \Gamma(\Theta)$.*

Suppose $(G \cap C) \in M \in \Theta$. Note that since $G \cap C \in \bigcup \Theta$, $\chi(C) \in G$, by definition of χ.

If ϕ denotes the identity function on C, then the one-element range refinement $\phi_{+\chi(C)}$ (defined above) copies $G \cap C$ to $G \cap C^{+\chi(C)}$; therefore $G \cap C^{+\chi(C)} \in M$ by Lemma 4.5.

$\chi(C^{+\chi(C)}) = \chi(C)$ by Lemma 6.4, so $\chi(C^{+\chi(C)}) \in G \cap C^{+\chi(C)}$; thus $(G \cap C^{+\chi(C)}) \in \Gamma_0(\Theta)$.

Note this holds for all C, M as in the lemma's statement.

Now suppose there is a least ordinal $\delta > 0$ such that *some* C with $G \cap C \in \bigcup \Theta$ satisfies $(G \cap C^{+\chi(C)}) \notin \Gamma_\delta(\Theta)$, and fix one C witnessing this. Note δ must be a successor ordinal since otherwise $\Gamma_\delta(\Theta) = \bigcap_{\beta<\delta} \Gamma_\beta(\Theta)$, but $(G \cap C^{+\chi(C)}) \in \Gamma_\beta(\Theta)$ for all $\beta < \delta$.

Suppose that D, S, M witness $C^{+\chi(C)}$'s violation of the definition of generic refinability in $(\Gamma_{\delta-1}(\Theta), \Theta)$. Fix any $s \in S \cap G$ such that $s \leq \chi(D)$, if $\chi(D) \in G$, or otherwise such that $s \leq \neg\chi(D)$. (Such an s exists because G is generic.) We now need to find $N \in \Theta$ and $F' \in N$ that together with s will satisfy the condition of definition of generic refinability for D, S, M, contradicting the supposed violation of that definition.

Let E be such that $G \cap E \in M$ (some such E exists by Lemma 4.5). $G \cap E \in \bigcup \Theta$ implies by definition of χ that $\chi(E) \in G$. Let E' denote $E^{+\chi(E)+s}$. By Lemma 6.4, $\chi(E') = \chi(E)$, so $\chi(E') \in G$.

We consider two cases.

If $\neg\chi(D) \in G$, let E'' denote $\overline{C^{+\chi(C)} \cup E'}$; then clause (ii) of Lemma 4.2 on Θ, plus the fact that the meet of any pair of G-members is a G-member, entails $\chi(E'') \in G$. So by definition of χ, $G \cap E'' \in \bigcup \Theta$. Let $F'' = G \cap E''$ and let N be the unique Θ-member such that $F'' \in N$. By the leastness of δ, $F'' \in \Gamma_{\delta-1}(\Theta)$. So our F'', N satisfy the generic-refinability requirement for $C^{+\chi(C)} \in \Gamma_\delta(\Theta)$, contradiction.

If $\chi(D) \in G$, we argue similarly, but with E'' defined as $\overline{C^{+\chi(C)} \cup D \cup E'}$. We know that the χ-values of the three subalgebras used to define E'' are all in G; so as before we get $\chi(E'') \in G$, entailing $G \cap E'' \in \bigcup \Theta$. Letting $F'' = G \cap E''$ again, the same conclusions about F'' follow, and in addition we have that $F'' \cap D$ is a generic filter on D, fulfilling the last clause of generic refinability. So again F'', N satisfy the generic-refinability requirement for $C^{+\chi(C)} \in \Gamma_\delta(\Theta)$.

Thus no such ordinal δ can exist, and $C^{+\chi(C)} \in \Gamma(\Theta)$. □

Theorem 6.8 (Self-collection condition in terms of filters) *Under Stipulation 3.10, $G \in L(\Theta)$ iff there exists $D \in \mathsf{ACSAs}(B)$ such that $(G \cap D) \in \Gamma(\Theta)$ and all $(G \cap D)$'s refinements in $\Gamma(\Theta)$ are mutually compatible, in which case the proposition "$x \in F \in \Gamma(\Theta)$ for some refinement F of $G \cap D$" serves as the predicate $\Psi(\dots)$ for Lemma 5.1.*

For $D \in \mathsf{ACSAs}(B)$ satisfying $(G \cap D) \in \Gamma(\Theta)$, let Q_D be the set of all fully-refinable refinements of $G \cap D$, that is:

$$Q_D \equiv \{F \in \Gamma(\Theta) : G \cap D \subseteq F\}.$$

Since $\Gamma(\Theta) \in L(\Theta)$ (Lemma 6.6), $Q_D \in L(\Theta)$, and $\bigcup Q_D \in L(\Theta)$.

We begin with the "if" direction. If D exists as in the lemma's statement, then all Q_D-members are mutually compatible; we will show this entails $G = \bigcup Q_D$, which suffices. Note first that by Lemma 6.7, $G \cap D^{+\chi(D)}$ is a refinement of $G \cap D$ that is a member of $\Gamma(\Theta)$. Now for any $b \in B$, consider $(D^{+\chi(D)})^{+b}$. By Lemma 6.4, we have

$$\chi((D^{+\chi(D)})^{+b}) = \chi(D^{+\chi(D)}) = \chi(D);$$

thus $(D^{+\chi(D)})^{+b}$ already has its own χ-value as a member, and so again by Lemma 6.7, $(G \cap (D^{+\chi(D)})^{+b})$ is a refinement of $G \cap D$ in $\Gamma(\Theta)$, and so is a Q_D-member. So $b \in G$ iff $b \in \bigcup Q_D$, and since this holds for all b, $G = \bigcup Q_D$.

Next we prove the "only if" direction; supposing there is no D meeting the condition stated in the lemma, we will show that Θ does not construct G. Suppose towards a contradiction that Θ *did* construct G, and let Ψ, C, and $\vec{c}$ witness this in Lemma 5.1. Some $g \in G$ must force them to be witnesses; so let us choose $g \in G, g \leq \chi(C)$ such that

$$g \Vdash \{x : \Psi(x, \dot{\Theta}, \dot{G} \cap \check{C}, \vec{c})\} = \dot{G}.$$

We will now show that our suppositions permit the existence of a G' contradicting Lemma 5.2.

Let D be $(C^{+g})^{+\chi(C)}$. Note that $G \cap D \in \Gamma(\Theta)$ by Lemma 6.7. We will obtain our G' by forcing over the set Q_D ordered by reverse inclusion. Note that $L(G)$ is the ground model for this forcing. Let Y be a generic filter obtained by this forcing, and define G' to be $\bigcup Y$.

G' is clearly a filter on B^+. It is generic over L by the following argument. Let $S \in L$ be any dense subset of B^+. Because each $F \in Q_D$ satisfies $F \in \Gamma(\Theta)$, and each of its refinements in $\Gamma(\Theta)$ is in Q_D, any $F \in Y$ can be refined (by Lemma 6.5) to $F' \in Q_D$ with $F \cap S \neq \emptyset$. Thus, by genericity, some such F' is in Y, and $G' \cap S \neq \emptyset$.

It follows immediately from the definitions of D and of Q_D that $G \cap D = G' \cap D$; and since $g \in G \cap D$, also $g \in G' \cap D$.

To show that $G' \neq G$, note every $F \in Q_D$ can be refined to some $F' \in Q_D$ that is incompatible with G. For if some F were not so refinable, it would equal $G \cap D'$ for some $D' \supseteq D$, and all its refinements in $\Gamma(\Theta)$ would be mutually compatible (being compatible with G); D' would thus meet the requirements to be the D that we are supposing not to exist. By genericity/density, some F' inconsistent with G must therefore be in Y, whence $G' \neq G$.

To see that G' gives the same interpretation as G to $\dot{\Theta}$, consider first any unconstructible $M \in (\dot{\Theta})_G$. (For the remainder of this argument we use the notation $(\dot{\Theta})_G$ for G's interpretation of $\dot{\Theta}$, which we had been calling Θ, in order to avoid confusing it with $(\dot{\Theta})_{G'}$.) We will show that also $M \in (\dot{\Theta})_{G'}$.

Let $F \in Q_D$ be arbitrary. Since $F \in \Gamma(\Theta)$, there exists by Lemma 6.5 and the definition of generic refinability some $N \in (\dot{\Theta})_G$, and $F' \in N \cap \Gamma(\Theta)$, such that F' refines F and $M \in L(N)$. Since any $F \in Q_D$ is refinable to such an F', some such F' must, by Y's genericity, be a member of Y. Let $E \in \mathsf{ACSAs}(B)$ be the subalgebra on which F' is an ultrafilter. The definition of χ-accordance of filters along with $G' = \bigcup Y$ gives us $\chi(E) \in G'$; thus F' is a member of $\bigcup(\dot{\Theta})_{G'}$. Now since $M \in L(N)$, it follows from Lemma 4.9, the definition of Θ, and Lemma 4.5 that there must be an ACSA $E' \subseteq E$ such that $G' \cap E' \in M$. Lemma 4.4 applied to $(\dot{\Theta})_{G'}$ then entails that $G' \cap E'$ is a member of $\bigcup(\dot{\Theta})_{G'}$, so $M \in (\dot{\Theta})_{G'}$.

Conversely, fix an unconstructible $M \in (\dot{\Theta})_{G'}$. Since G', like G, is an L-generic ultrafilter on B, Lemma 6.7 applies to it, ensuring the existence of an E such that $G' \cap E \in M$. Now it would suffice to show that there exists an $F' \in Y$ such that F' refines $G' \cap E$, because then:

$F' \in Y \Rightarrow$

$F' \in Q_D \Rightarrow$

$F' \in \bigcup(\dot{\Theta})_G \Rightarrow$

$F' \in N \in (\dot{\Theta})_G$ for some $N \Rightarrow$

F' is a copy of some $G \cap A \in N$ via some $\phi \in \mathsf{ParAut}(B)$ (Lemma 4.5) $\Rightarrow$

$G' \cap E$ is a copy via (a restriction of) ϕ of $G \cap A'$ for some $A' \subseteq A \Rightarrow$

$G' \cap E \in \bigcup(\dot{\Theta})_G$ (Lemma 4.4) $\Rightarrow$

$M \in (\dot{\Theta})_G$.

To see that such an $F' \in Y$ does exist, consider that by the definition of generic refinability, any $F \in Q_D$ can be refined to some $F' \in Q_D$ such that either $\neg\chi(E) \in F'$, or $F' \cap E$ is a generic filter on E. By Y's genericity, then, some such F' is in Y, and since we already know $\chi(E) \in \bigcup Y$, the second alternative must hold, which entails that F' refines $G' \cap E$. □

7 Self-Collection Condition In Terms Of Isomorphisms

The preceding theorem gives a self-collection condition that can be checked in the generic extension, stated in terms of generic filters; we now prove lemmas that will let us translate this condition into a condition on the boolean algebra B (given in Theorem 7.7) that can be checked in the ground model, stated in terms of $\mathsf{ParAut}(B)$.

Definition. $\phi \in \mathsf{ParAut}(\mathsf{B})$ *fixes* $C \in \mathsf{ACSAs}(B)$ if it is a refinement of the identity function on C. This definition is equivalent to:

$$\max(\operatorname{dom}\phi) \wedge [C] \subseteq \operatorname{dom}\phi, \text{ and } (\forall b \in \operatorname{dom}\phi)(\phi(b) \uparrow C = b \uparrow C).$$

Lemma 7.1 *If ϕ copies $G \cap C$ to F and $b \in G$, ϕ^{+b} copies $G \cap C^{+b}$ to F.*

(We assume ϕ^{+b} is well defined, so $b \leq \max(\operatorname{dom}\phi)$.) Let D denote the ACSA on which F is an ultrafilter. We must show that for all $d \in D$,

$$(*) \quad (\exists c \in G \cap \operatorname{dom}\phi)(\phi(c) \leq d) \iff (\exists c' \in G \cap \operatorname{dom}\phi^{+b})(\phi^{+b}(c') \leq d).$$

We showed in Lemmas 6.2 and 6.3 that the map $\psi : c \mapsto b \wedge c$ defined on $C \restriction (b \uparrow C)$ is an isomorphism, and that $\phi^{+b}(c') = \phi(\psi^{-1}(c'))$ for all $c' \in \operatorname{dom}\phi^{+b}$. Since G is an ultrafilter on B and $b \in G$, we have for all $c \in \operatorname{dom}\phi$ that $c \in G \iff c \wedge b \in G$. Now every $c' \in \operatorname{dom}\phi^{+b}$ is of form $c \wedge b$ for some $c \in \operatorname{dom}\phi$, from which (*) follows. □

Lemma 7.2 *If ϕ copies $G \cap C$ to F, and F' refines F, and $F' \in \theta(Y)$ for some singly-generated continuum $Y \in L[G]$, there exist ϕ' and $D' \supseteq C$ such that ϕ' copies $G \cap D'$ to F' and ϕ' refines ϕ.*

By Lemma 4.5, *some* $\gamma \in \mathsf{ParAut}(B)$ copies *some* $G \cap C'$ to F'. Let $E \subseteq E'$ be the ACSAs on which F, F' are generic ultrafilters respectively. Let $D \subseteq C'$ be an ACSA of which $\gamma^{-1}[E \cap \operatorname{ran} \gamma]$ is a principal ideal, so that $D \cap G$ is interconstructible with $F' \cap E = F$. Thus $D =_G C$.

By Lemma 4.9 applied "in both directions," there exist $g_1, g_2 \in G$ such that $g_1 \wedge [D] \subseteq g_1 \wedge [C]$ and $g_2 \wedge [C] \subseteq g_2 \wedge [D]$. Since G is a filter, $g \equiv g_1 \wedge g_2 \in G$, and $g \wedge [C] = g \wedge [D]$.

Define $\gamma' \equiv \gamma^{+g}$, the one-element refinement of γ by g. If we define $D' \equiv \overline{C \cup C'}$, it is easy to check that $\operatorname{dom} \gamma'$ is a principal ideal of D', and that γ' refines ϕ, and copies $G \cap D'$ to F'. □

Corollary 7.3 *If $F' \in \theta(Y)$ for some singly-generated continuum Y, and $G \cap C \subseteq F'$ for some $C \in \mathsf{ACSAs}(B)$, then there exists $D' \supseteq C$ and $\phi' \in \mathsf{ParAut}(B)$ such that ϕ' copies $G \cap D'$ to F', and ϕ' fixes C.*

In Lemma 7.2, let F be $G \cap C$, and let ϕ be the identity function on C. □

Lemma 7.4 *In light of Stipulation 3.10's consequences for χ (Lemma 4.3), every $F \in \Gamma(\Theta)$ is the copy of some $G \cap E$ via some $\phi \in \Phi$.*

In $L[G]$, by Lemma 4.5, every generic filter F on any $D \in \mathsf{ACSAs}(B)$ is a copy via some $\phi \in \mathsf{ParAut}(B)$ of $G \cap E$ for some E. We must show that when $F \in \Gamma(\Theta)$, the ϕ and E can be chosen so that $\phi \in \Phi$. In fact we will show something stronger: given any choice of ϕ, E, there exists b such that $\phi^{+b} \in \Phi$ and ϕ^{+b} copies $G \cap E^{+b}$ to F.

Let $\Delta(\alpha)$ be the following assertion about an ordinal α:

$\Delta(\alpha) \iff$ *For every $F \in \Gamma(\Theta)$ and every ϕ copying some $G \cap E$ to F, there exists $b \in G$, $b \leq \max(\operatorname{dom} \phi)$, such that $\phi^{+b} \in \Phi_\alpha$.*

Any such ϕ^{+b} will copy $G \cap E^{+b}$ to F, by Lemma 7.1. Thus our "stronger" claim above will follow if $\Delta(\alpha)$ holds for all α (since $\Phi = \Phi_\alpha$ for some α). Let us suppose towards a contradiction that $\Delta(\alpha)$ fails for some least α. Fix F, ϕ, and E witnessing the failure, and let D be the ACSA on which F is an ultrafilter. There are then three cases.

Case I: $\alpha = 0$. Since $F \in \Gamma_0(\Theta)$, it accords with χ, so $\chi(D)$ is in F's upwards closure in B. Choose some $e \in \phi[G \cap \operatorname{dom} \phi]$ with $e \leq \chi(D)$. Let $b = \phi^{-1}(e)$. Then $\phi \restriction b = \phi^{+b}$, and $b \in G$. Now since $F \in \bigcup \Theta$, $G \cap E \in \bigcup \Theta$ and so $\chi(E) \in G$; thus

$g \equiv \chi(E) \wedge b$ is in G, since G is a filter. It is then easy to check that ϕ^{+g} accords with χ, so $\phi^{+g} \in \Phi_0$. This contradicts the supposition that F, ϕ, E witness $\neg\Delta(0)$.

Case II: α is a successor ordinal. Since F, ϕ, and E supposedly witness $\neg\Delta(\alpha)$, we have for all $b \in G$ with $b \leq \max(\mathrm{dom}\,\phi)$ that $\phi^{+b} \notin \Phi_\alpha$. By the leastness of α, however, there does exist such a b satisfying $\phi^{+b} \in \Phi_{\alpha-1}$. Indeed, whenever $b \in G$ and $b \leq \max(\mathrm{dom}\,\phi)$, ϕ^{+b} will copy $G \cap E^{+b}$ to F (by Lemma 7.1), and $\Delta(\alpha-1)$ will hold for F, ϕ^{+b}, and E^{+b} too, so that there will exist $b' \leq b$ satisfying $\phi^{+b'} \in \Phi_{\alpha-1}$. For all such b', $\phi^{+b'}$ is in $\Phi_{\alpha-1}$ but fails to be generically refinable within $\Phi_{\alpha-1}$, and this failure must be witnessed by some C, D, S as in the definition of generic refinability (for isomorphisms); in particular $b' \wedge \chi(C) \neq 0$. Now by genericity we must have $b' \wedge \chi(C) \in G$ for some such b' having associated witnesses C, D, S (note this implies $\chi(C) \in G$). Fix such a b' and associated C, D, S.

Now define $C' \equiv \overline{C \cup E}$. Because $\chi(C), \chi(E) \in G$, stipulation (ii) on χ (from Lemma 4.3) entails $\chi(C') \in G$. Therefore there exists $M \in \Theta$ such that $G \cap C' \in M$. Since $F \in \Gamma(\Theta)$, Lemma 6.5 ensures that it refines generically in $(\Gamma(\Theta), \Theta)$. In particular (reading off the definition of generic refinability for filters) F refines to some $F' \in N \in \Theta$, for some N with $M \in L(N)$, such that $F' \cap S \neq \emptyset$, and $F' \in \Gamma(\Theta)$, and either $\neg\chi(D) \in F'$ or $F' \cap D$ is an L-generic filter on D.

Now by Lemma 7.2, this $F' \in \Gamma(\Theta)$ is a copy of $G \cap C''$ for some $C'' \supseteq C'$, via some ϕ' that refines ϕ. We know $\Delta(\alpha - 1)$ holds, so some ϕ'^{+e} with $e \in G$ and $e \leq \max(\mathrm{dom}\,\phi')$ satisfies $\phi'^{+e} \in \Phi_{\alpha-1}$. By Lemma 7.1, ϕ'^{+e} copies $C''^{+e} \cap G$ to F'. Consider the restriction ϕ'' of ϕ'^{+e} to some principal ideal of C''^{+e} such that max(ran ϕ'') is $\leq$ some member of $F' \cap S$. It is then straightforward to check that ϕ'' is a refinement of $\phi^{+b'}$ that satisfies the requirements of generic refinability within $\Phi_{\alpha-1}$ for our fixed C, D and S, contradicting our choices.

Case III: α is a limit ordinal. The argument here is similar. Recall that for limit α, Φ_α is just the intersection of the Φ_β's with $\beta < \alpha$. Recall our fixed F, ϕ, and E witnessing $\neg\Delta(\alpha)$: again, for all $b \in G$ with $b \leq \max(\mathrm{dom}\,\phi)$, $\phi^{+b} \notin \Phi_\alpha$. But now since α is a limit ordinal, this means for all such b there exists $\gamma < \alpha$, and thus a *least* $\gamma < \alpha$, such that $\phi^{+b} \notin \Phi_\gamma$; moreover this γ must be a successor ordinal. As before, for each such b, some C, D, S will witness $\phi^{+b} \in \Phi_{\gamma-1} \setminus \Phi_\gamma$ (for b's particular γ), so that in particular $b \wedge \chi(C) \neq 0$. As before, we have by genericity that $b \wedge \chi(C) \in G$ for some such b with its associated C, D, S. The contradiction is then obtained just as before, but with γ in place of α. $\square$

Lemma 7.5 *If the F in Lemma 7.4 satisfies $G \cap D \subseteq F$ for some D, then the E and ϕ can be chosen such that ϕ fixes D.*

The proof of Lemma 7.4 showed that, given any choice of ϕ, E such that ϕ copies $G \cap E$ to F, there exists b such that $\phi^{+b} \in \Phi$ and ϕ^{+b} copies $G \cap E^{+b}$ to F. By

Corollary 7.3, we may choose a ϕ that fixes D (is a refinement of the identity function on D), and the resulting ϕ^{+b} will also be a refinement of the identity on D. □

Lemma 7.6 *If $\phi \in \Phi$, $\max(\operatorname{dom}\phi) \in G$, and ϕ copies $(G \cap \operatorname{dom}\phi)$ to F, then $F \in \Gamma(\Theta)$.*

Note this is a converse to Lemma 7.4; in this direction, we will argue much the same way. Let $\Psi(\alpha)$ be the following assertion about an ordinal α:

$\Psi(\alpha) \iff$ *every filter F that is a copy of $G \cap E$ for some E, via some $\phi \in \Phi$, is a member of $\Gamma_\alpha(\Theta)$.*

Suppose toward a contradiction that $\Psi(\alpha)$ fails for some least α. Fix F, E, ϕ witnessing this failure, and let D denote the ACSA on which F is an ultrafilter.

Case I: $\alpha = 0$. Since ϕ accords with χ, $\max(\operatorname{ran}\phi) \leq \chi(D)$. Since $\max(\operatorname{dom}\phi) \in G$, $\max(\operatorname{ran}\phi) \in \phi[G \cap \operatorname{dom}\phi] \subseteq F$, so $\chi(D)$ is in the upward closure of F, which is just the definition of accordance with χ for F. Also $F \in \bigcup\Theta$ follows from Lemma 4.5 because $G \cap E \in \bigcup\Theta$ (since $\max(\operatorname{dom}\phi) \in G$ and $\max(\operatorname{dom}\phi) \leq \chi(E)$). So $F \in \Gamma_0(\Theta)$ after all.

Case II: α is a successor ordinal. By α's leastness, $F \in \Gamma_{\alpha-1}(\Theta)$; but we have supposed $F \notin \Gamma_\alpha(\Theta)$, so F fails to refine generically in $(\Gamma_{\alpha-1}(\Theta), \Theta)$. Let some D, S, M witness this failure, as in the definition of generic refinability of filters. In particular note that $M \in \Theta$; since also $G \cap E \in \bigcup\Theta$, there must exist by the directedness of $\bigcup\Theta$ (stipulation (ii) on Θ from Lemma 4.2) some $N \in \Theta$ and $C \in \mathsf{ACSAs}(B)$ such that $E \subseteq C$, and $G \cap C \in N$, and $M \in L(N)$. Fix such an N and C. Since $G \cap C \in \bigcup\Theta$, $\chi(C) \in G$, so $\chi(C)$ is compatible with $\max(\operatorname{dom}\phi)$ (which is also in G). Now apply the definition of ϕ's being generically refinable within Φ, using our present C, D and S, to obtain $s \in S$ and $\phi' \in \Phi$ refining ϕ, meeting the several requirements of that definition.

Let $A \in \mathsf{ACSAs}(B)$ have $\operatorname{ran}\phi'$ as a principal ideal. Then A^{+s} also has it as a principal ideal, because $\max(\operatorname{ran}\phi') \leq s$. Now consider the two cases in the final clause of the definition that was just invoked to obtain ϕ'. If $\max(\operatorname{ran}\phi) \leq \neg\chi(D)$, define

$$A' \equiv (A^{+s})^{+\neg\chi(D)};$$

otherwise we define

$$A' \equiv \overline{A^{+s} \cup D}.$$

In both cases we still have $\operatorname{ran}\phi$ a principal ideal of A''; in the latter case this is because $\max(\operatorname{ran}\phi') \wedge [D] \subseteq \operatorname{ran}\phi'$.

Now note that $F' \in \Gamma_{\alpha-1}(\Theta)$, by the way F' was obtained as a copy via $\phi' \in \Phi$ and because $\Psi(\alpha-1)$ holds. It is then straightforward to verify, by $F' \in \Gamma_{\alpha-1}(\Theta)$, by the requirements that ϕ' meets, and by the definition of A'', that our N, F', and s meet all the requirements of the generic refinability definition for our D, S, M — the requirements that we supposed unsatisfiable.

Case III: There is no Case III here because the least α at which $\Psi(\alpha)$ fails cannot be a limit ordinal: if every filter F of the kind specified in the definition of $\Psi(\alpha)$ is a member of $\Gamma_\beta(\Theta)$ for all $\beta < \alpha$, then $F \in \Gamma_\alpha(\Theta)$, since the latter is just the intersection of all the $\Gamma_\beta(\Theta)$ with $\beta < \alpha$. □

Theorem 7.7 *Let X be the set of all $d \in B$ for which there exists $D \in \mathsf{ACSAs}(B)$, $d \le \chi(D)$, such that every $\phi \in \Phi$ that fixes D and satisfies* $\max(\operatorname{dom}\phi) \wedge d > 0$ *is an identity mapping. Then under Stipulation 3.10, $\|G \in L(\Theta)\| = \bigvee X$.*

It suffices to show that there exists a $d \in X \cap G$ just if a D meeting the condition of Theorem 6.8 exists.

Suppose D witnesses $d \in X$, for some $d \in G$. Thus we have $d \le \chi(D)$, so that $\chi(D) \in G$ and $D \cap G \in \bigcup \Theta$, and every $\phi \in \Phi$ that fixes D and satisfies $\max(\operatorname{dom}\phi) \wedge d > 0$ is an identity mapping. We claim that $D^{+\chi(D)}$ satisfies the condition of Theorem 6.8. We must show (reading off that condition) that $G \cap D^{+\chi(D)} \in \Gamma(\Theta)$ and all refinements $F \in \Gamma(\Theta)$ of $G \cap D^{+\chi(D)}$ are mutually compatible. The first requirement holds by Lemma 6.7. For the second, suppose that some $F, F' \in \Gamma(\Theta)$ violated it and (without loss of generality) that F is not compatible with G. By Lemma 7.4, F would have to be a copy, via some non-identity $\phi \in \Phi$, of $G \cap E$, for some E. By Lemma 7.5, we can also require that ϕ fix $D^{+\chi(D)}$, and hence D. Since ϕ copies $G \cap E$, we have $\max(\operatorname{dom}\phi) \in G$; thus $\max(\operatorname{dom}\phi) \wedge d$ belongs to G and so must be > 0. This ϕ contradicts our assumption that D witnessed $d \in X$.

Similarly, in the other direction, suppose D satisfying $(G \cap D) \in \Gamma(\Theta)$ meets the condition of Theorem 6.8: every $F \in \Gamma(\Theta)$ that refines $G \cap D$ is of form $G \cap E$ for some $E \supseteq D$. Choose some $d \in G$ that forces this condition to hold. We claim that D witnesses $d \in X$. If not, some non-identity $\phi \in \Phi$ fixes D and satisfies $\max(\operatorname{dom}\phi) \wedge d > 0$. But by Lemma 7.6, if we had $\max(\operatorname{dom}\phi) \in G$, ϕ would copy $(G \cap \operatorname{dom}\phi)$ to a filter $F \in \Gamma(\Theta)$. Since ϕ is not an identity mapping, this F is not of form $G \cap E$. So $\max(\operatorname{dom}\phi) \wedge d$ would force this F to violate the condition that d supposedly forced. □

8 Negative Results On Self-Construction

We will now establish that the self-collection criteria derived in the preceding sections are not trivial to satisfy. In particular, we will show that any $\mathcal{N}$ meeting Stipulation 3.10 fails to self-collect into $\mathbb{R}(G)$ if B has a "flexible homogeneity" property ensuring that it has certain partial automorphisms. Since this property makes no mention of $\mathcal{N}$ (nor of any correlatively-defined structures like Θ or χ), one upshot is that, even assuming B yields a rich structure of continua when used for forcing, it may just have too many partial automorphisms, so that no amount of cleverness in designing a B-name for $\mathcal{N}$ can suffice to make $\mathcal{N}$ self-collect into $\mathbb{R}(G)$. We will show that this is the case when B is a Cohen-real or random-real algebra.

8.1 A property that precludes self-collection

We now begin arguments culminating with Theorem 8.6, to show that when our boolean algebra B has the "flexible homogeneity" property defined just below, B cannot yield singly-generated continua that self-collect into another such continuum.

Definitions:

The definitions and stipulations of Sections 2, 3, and 4 remain in force (for B, G, $\mathsf{ACSAs}(B)$, $\mathcal{N}$, Θ, χ, etc.).

B is *homogeneous* if it is isomorphic to each of its principal ideals.

B is *ACSA-homogeneous* if it is isomorphic to every principal ideal of every $C \in \mathsf{ACSAs}(B)$.

B is *flexibly homogeneous* if:

(1) B is ACSA-homogeneous;

(2) there exists $C \in \mathsf{ACSAs}(B)$ satisfying $1 \Vdash C \neq_G B$;

(3) for each C satisfying (2) there exists an automorphism on B that is not an identity mapping but whose restriction to C is an identity mapping;

(4) each isomorphism between any C, C' satisfying (2) extends to an automorphism of B.

We will also apply the above terms to a principal ideal $B \restriction b$ if it meets the relevant definitions when considered as a boolean algebra in its own right (with greatest element b and negation operation derived from B's negation as $x \mapsto b \wedge \neg x$).

Lemma 8.1 *If B is flexibly homogeneous, then each principal ideal of each $C \in$ $\mathsf{ACSAs}(B)$ (including the "improper ideal," C itself) is also flexibly homogeneous.*

Immediate from clause (1) of flexible homogeneity. □

Lemma 8.2 *Given $C \in \mathsf{ACSAs}(B)$ and $b \in B^+$ such that $b \leq \chi(C)$, and assuming that $\|\mathcal{N}$ self-collects into $\mathbb{R}(G)\| = 1$, there exists $E \supset C$ such that*

$$b \wedge \chi(E) \wedge \|E >_G C\| > 0,$$

and each term in that conjunction is a member of E.

The main point here is that no $\mathbb{R}(G \cap C)$ can be a maximal member of $\bigcup \mathcal{N}$; there must exist a larger $\mathbb{R}(G \cap E) \in \bigcup \mathcal{N}$, and we justify some additional demands on E for technical simplification later.

The key is to show that some D must satisfy

$$b \wedge \chi(D) \wedge \|D \not\leq_G C\| > 0.$$

Suppose not. Then for all D,

$$(b \wedge \chi(D)) \Vdash D \leq_G C.$$

From this and the definitions of χ and Θ it follows that

$$b \Vdash [(\mathbb{R}(G \cap D) \subseteq \bigcup \mathcal{N}) \Rightarrow (\mathbb{R}(G \cap D) \subseteq \mathbb{R}(G \cap C)],$$

and since that holds for all D, we have

$$b \Vdash \mathbb{R}(\bigcup \mathcal{N}) = \mathbb{R}(G \cap C).$$

From this and $\|\mathcal{N}$ self-collects into $\mathbb{R}(G)\| = 1$ it follows that

$$b \Vdash \mathbb{R}(G \cap C) = \mathbb{R}(G).$$

Now since $b \leq \chi(C)$, it follows from the definitions of Θ and χ in terms of $\mathcal{N}$ that

$$b \Vdash (\exists C')(\mathbb{R}(G \cap C) \subseteq \mathbb{R}(G \cap C') \text{ and } \mathbb{R}(G \cap C') \in \mathcal{N}),$$

which along with the previous step gives us

$$b \Vdash \mathbb{R}(G) \in \mathcal{N},$$

which would violate clause (i) of the definition of self-collection.

The foregoing permits us to fix an ACSA D such that when we define

$$e \equiv b \wedge \chi(D) \wedge \|D \not\leq_G C\|,$$

we have $e > 0$.

Now define $E = \overline{C \cup D}$. Plainly $C \subseteq E$, $C \leq_G E$, and $D \leq_G E$. Since $e \leq \chi(D)$, and $e \leq b \leq \chi(C)$, we have by (ii) of Lemma 4.3 that $e \leq \chi(E)$. We claim that $e \leq \|E >_G C\|$. This follows from the fact (Corollary 3.3) that $<_G$ is guaranteed with boolean value 1 to be a preorder: if some $e' \leq e$ forced $E =_G C$ and we had $e' \in G$, then in $L[G]$ we would have $E =_G C$ and $D \leq_G E$, entailing $D \leq_G C$, contradicting our choice of D.

Thus E meets all our requirements except possibly the requirement that b, $\chi(E)$ and $\|E >_G C\|$ be members. To ensure this, simply replace E with $E^{+b+\chi(E)+\|E>_G C\|}$; Lemma 6.4 ensures that E's χ-value stays the same, and clearly the other requirements on E in the statement of the lemma will still hold when E is enlarged. □

Lemma 8.3 *Suppose B is flexibly homogeneous; if $C \in \mathsf{ACSAs}(B)$ and $g \in G$ were witnesses to $\mathcal{N}$'s self-collection into $\mathbb{R}(G)$ as in Theorem 7.7, then there would exist $\phi \in \Phi_0$ with $\max(\operatorname{dom}\phi) \leq g$ that fixes C but is not an identity function.*

Define $g' \equiv g \wedge \|\mathcal{N}$ self-collects into $\mathbb{R}(G)\|$. Note the fact that C, g witness Theorem 7.7 entails $g \leq \chi(C)$, so we also have $g' \leq \chi(C)$.

Consider $B \restriction g'$ and $C^{+g'} \restriction g'$. By Lemma 8.2, there exists $E \in \mathsf{ACSAs}(B \restriction g')$ such that $E \supset (C^{+g'} \restriction g')$, and when we define

$$e \equiv \chi(E) \wedge \|E >_G (C^{+g'} \restriction g')\|,$$

we have $0 < e \leq g'$, and also $e \in E$.

Now by clause (3) of flexible homogeneity applied to $C^{+e} \restriction e$ considered as an ACSA of $E \restriction e$, there exists an automorphism ϕ of $E \restriction e$ whose restriction to $C^{+e} \restriction e$ is the identity. And we have $\phi \in \Phi_0$ because $\max(\operatorname{ran}\phi) = \max(\operatorname{dom}\phi) = e \leq E'$, where E' is any ACSA of (all of) B such that $E' \restriction e = E$. □.

Lemma 8.4 *If $\Phi_1 = \Phi_0$ then $\Phi = \Phi_0$.*

Suppose $\Phi_1 = \Phi_0$ but there is a least ordinal α (necessarily > 1) such that there exists $\phi \in \Phi_0 \setminus \Phi_\alpha$. Note α cannot be a limit ordinal since in that case Φ_α would be the intersection of the Φ_β for all $\beta < \alpha$. Thus there exists some $\phi \in \Phi_{\alpha-1}$ that fails to refine generically within $\Phi_{\alpha-1}$. However $\Phi_{\alpha-1} = \Phi_0$ and we already know ϕ refines generically within Φ_0, since $\phi \in \Phi_1$. □

Lemma 8.5 *Suppose B is flexibly homogeneous; if $\|\mathcal{N}$ self-collects into $\mathbb{R}(G)\|$ were equal to 1, then $\Phi = \Phi_0$.*

By Lemma 8.4 it suffices to show that any $\phi \in \Phi_0$ satisfies $\phi \in \Phi_1$ — equivalently, refines generically within Φ_0. So fix $\phi \in \Phi_0$, and, for the parameters of the definition of generic refinability, fix an arbitrary dense subset $S \subseteq B^+$ and $C, D \in \mathsf{ACSAs}(B)$, such that when we define

$$a \equiv \chi(C) \wedge \max(\operatorname{dom}\phi) > 0,$$

we have $a > 0$. We will show that the required ϕ' exists.

Let $C_0 \in \mathsf{ACSAs}(B)$ be such that $\operatorname{dom}\phi$ is a principal ideal of C_0, and likewise for D_0 with $\operatorname{ran}\phi$. (In general, variable names involving "C" will denote ACSA's involved with the domain of the ϕ' we are seeking, and likewise with "D" and range of ϕ'.) Note that since $\phi \in \Phi_0$, we have $\max(\operatorname{dom}\phi) \leq \chi(C_0)$ and $\max(\operatorname{ran}\phi) \leq \chi(D_0)$.

Define $C_1 \equiv \overline{C_0 \cup C}$. By clause (ii) of Lemma 4.3, $\chi(C_1) = \chi(C_0) \wedge \chi(C)$. Therefore we have $a \leq \chi(C_1)$.

By Lemma 8.2, there exists $E_C \supseteq C_1$ such that, when we set

$$c \equiv a \wedge \chi(E_C) \wedge \|E_C >_G C_1\|,$$

we have $c > 0$, and each conjunct in c's definition is a member of E_C.

Now ϕ^{+c}, the one-element domain refinement of ϕ by c, is well-defined because $c \leq a \leq \max(\operatorname{dom}\phi)$. Note $\max(\operatorname{dom}\phi^{+c}) = c$, and

$$\max(\operatorname{ran}\phi^{+c}) \leq \max(\operatorname{ran}\phi) \leq \chi(D_0).$$

To fix the ACSA E_D of which $\operatorname{ran}\phi'$ will be a principal ideal, we consider two cases.

Case I: $(\chi(D) \wedge \max(\operatorname{ran}\phi^{+c})) > 0$.

In this case let s be any S-member $\leq (\chi(D) \wedge \max(\operatorname{ran}\phi^{+c}))$, and define

$$D_1 \equiv \overline{D_0 \cup D}.$$

By (ii) of Lemma 4.3, $\chi(D_1) = \chi(D_0) \wedge \chi(D)$. Also we have

$$s \leq \chi(D_1) = \chi(D_0) \wedge \chi(D),$$

because

$$s \leq \max(\operatorname{ran}\phi^{+c}) \leq \chi(D_0).$$

Case II: $\chi(D) \wedge \max(\operatorname{ran}\phi^{+c}) = 0$.

In this case let s be any S-member below max ran ϕ^{+c}, and let $D_1 = D_0$ (so again we have $s \leq \chi(D_1)$).

Conclusion of argument for both cases:

Now use Lemma 8.2 to obtain $E_D \supseteq D_1$ such that when we define

$$d \equiv s \wedge \chi(E_D) \wedge \|E_D >_G D_1\|,$$

we have $d > 0$, and each conjunct in d's definition is a member of E_D.

Consider the isomorphism $(\phi^{+c})_{+d}$. Let C'_0 denote its domain, and let D'_0 denote its range. Let c' denote $\max(\mathrm{dom}((\phi^{+c})_{+d}))$; note $C'_0 = c' \wedge [C_0]$.

Let $E'_C \equiv E_C \upharpoonright c'$, and $E'_D \equiv E_D \upharpoonright d$. We consider E'_C and E'_D as boolean algebras in their own right. It is easily verified that $C'_0 \in \mathsf{ACSAs}(E'_C)$ and $D'_0 \in \mathsf{ACSAs}(E'_D)$.

Since $C_1 \supseteq C_0$, $\|C_1 \geq_G C_0\| = 1$, so $c' \leq \|E_C >_G C_0\|$. Thus the boolean 1 of E'_C forces $E'_C >_G C'_0$.

Now E'_C is isomorphic to E'_D by ACSA-homogeneity of B; choose an isomorphism $\xi : E'_C \to E'_D$ witnessing this. Consider the map $\xi \circ ((\phi^{+c})_{+d})^{-1}$ with domain D'_0. Its range is some ACSA $E^\star$ of E'_D. We have $\|E^\star <_G E'_D\| = 1$ since C'_0 is the range of $((\phi^{+c})_{+d})^{-1}$, and the boolean 1 of E'_C forces $E'_C >_G C'_0$.

Consider the inverse of $\xi \circ ((\phi^{+c})_{+d})^{-1}$, which maps $E^\star$ onto D'_0. By clause (4) of flexible homogeneity, it extends to an automorphism ζ of E'_D. Now define $\phi' \equiv \zeta \circ \xi$ by composition. It is straightforward to check that ϕ' is the refinement of ϕ demanded by the definition of generic refinability for our chosen parameters S, C, D. $\square$

Theorem 8.6 *If $B \in L$ is a countably-completely-generated boolean algebra that is flexibly homogeneous, and G is a generic filter on B, and $x \in \mathbb{R}(G) \setminus \mathbb{R}(\emptyset)$, no set $\mathcal{N}$ of singly-generated continua in $L[G]$ self-collects into $\mathbb{R}(x)$, and thus no set of such continua self-constructs.*

Suppose $\mathcal{N}$ did self-collect into $\mathbb{R}(x)$. We will first "zoom in" (if necessary) from all of B to a principal ideal of some ACSA, in order to reduce our situation to the one analyzed in our previous sections.

By Lemma 3.8 let B' be an ACSA of B such that $\mathbb{R}(G \cap B') = \mathbb{R}(x)$. By Lemma 6.4 we may assume that $\|\mathcal{N}$ self-collects into $\mathbb{R}(G \cap B')\| \in B'$. We wish to consider B' as our "outermost" algebra, and refer to $G \cap B'$ as G'.

Note that in passing to this (possibly) smaller algebra B' we do not bereave ourselves of all forcing names for $\mathcal{N}$, since the "$\mathcal{N} \in L(X)$" clause of the definition of self-collection ensures $\mathcal{N} \in L(\mathbb{R}(x)) = L(G \cap B')$, so that there is a B'-name for $\mathcal{N}$. Thus we assume Θ and χ to be defined relative to B' and to this B'-name.

Finally, in order to have the simplifying situation that "$\mathcal{N}$ self-collects into $\mathbb{R}(G \cap B')$" is forced by 1 (rather than just by some $g \in G$), let us zoom in further to the principal ideal

$B'' \equiv B' \restriction \|\mathcal{N}$ self-collects into $\mathbb{R}(G \cap B')\|$,

considered as a boolean algebra in its own right.

Now suppose that $D \in \mathsf{ACSAs}(B'')$ and $g \in G'$ witnessed the self-collection of $\mathcal{N}$ into $\mathbb{R}(x)$ in fulfillment of Theorem 7.7. Consider the ACSA D^{+g} and its principal ideal $D^{+g} \restriction g$. By Lemma 8.3, there exists $\phi \in \Phi_0$ that fixes D^{+g}, and is not an identity function. By Lemma 8.5, $\phi \in \Phi$. The existence of such a ϕ contradicts Theorem 7.7. $\square$

8.2 Cohen forcing does not work

Theorem 8.7 *No set of singly-generated continua in a Cohen-forcing extension of L self-collects into a singly-generated continuum.*

The boolean completion of a Cohen-forcing poset satisfies the definition of flexible homogeneity, as we will now show by devoting one lemma to each of that definition's four clauses; the theorem just stated will then follow from Theorem 8.6.

Definitions. A *Cohen-forcing algebra* is a complete atomless boolean algebra with a countable dense subset. We will see that there is only one such algebra up to isomorphism. Let K denote it.

A *free boolean algebra on κ generators* is the boolean algebra F_κ having a subset $\{g_\alpha : \alpha < \kappa\}$ (called *free generators*) that (1) finitely generates F_κ in the sense that every F_κ-member is the output of some boolean operation on some finite set of g_α's, and (2) has "mutually independent" members in the sense that the meet of any finite number of g_α's and/or negations thereof will be nonzero (assuming of course that for no α are both g_α and $\neg g_\alpha$ among the conjuncts). Note that F_κ is a complete boolean algebra if and only if κ is finite.

These two defining properties easily entail that whenever A, B are free boolean algebras having sets A_0, B_0 of generators with the same cardinality, any bijection between A_0 and B_0 induces an isomorphism between A and B.

Lemma 8.8 *If K is a Cohen-forcing algebra and $X \subseteq K$ is a (possibly empty) set of free generators for a subalgebra X' that is dense in the complete subalgebra $\overline{X}$ that it completely generates (in K), and $\|\overline{X} \geq_G K\| = 0$, then there exists $Y \subseteq K \setminus X$, such that $X \cup Y$ is a set of free generators of a subalgebra of K that is dense therein.*

By definition K^+ has a countable dense subset $\{q_n : n \in \omega\}$. We will use this subset to iteratively fix K^+-members y_n so that the set $Y = \{y_n : n \in \omega\}$ will be as required by the lemma's statement. Let us use K_n to denote the subalgebra of K that will be completely generated by $X \cup \{y_i : i < n\}$. In the notation for one-element expansions from Section 6, we then have

$$K_0 = \overline{X};$$

$$K_1 = \overline{X}^{+y_0};$$
$$K_n = \overline{X}^{+y_0+y_1+\ldots+y_{n-1}}.$$

By repeated invocation of Lemma 6.4 we have $\|K_n =_G \overline{X}\| = 1$, and thus by transitivity (Corollary 3.3), $\|K_n \geq_G K\| = 0$, for all n.

Let our induction hypothesis be that the set $X \cup \{y_i : i < n\}$ is a set of free generators of a subalgebra that is dense in K_n. This clearly holds when $n = 0$.

We now begin the inductive defintions. At stage $n \geq 0$, let m be least such that no finite plus/minus meet from $X \cup \{y_i : i < n\}$ is $\leq q_m$. Some such q_m must exist, lest K_n^+ be dense in K^+, contradicting $\|K_n \geq_G K\| = 0$. Set $y_n^0 \equiv q_m$ (the superscript "0" here is a secondary index rather than an exponent). If $(y_n^0 \uparrow K_n) = 1$, then $y_n^0 \wedge k > 0$ for all $k \in K_n^+$; for if k were a counterexample, then $1 > \neg k \geq y_n^0$ would contradict $(y_n^0 \uparrow K_n)$'s definition as the *least* K_n-member that is $\geq y_n^0$. So in this case (namely $y_n^0 \uparrow K_n = 1$) we may simply set $y_n \equiv y_n^0$. Otherwise, consider that since $(y_n^0 \uparrow K_n) < 1$, we have $\neg(y_n^0 \uparrow K_n) > 0$. There must exist $k \in K^+$ such that $k \leq \neg(y_n^0 \uparrow K_n)$ but no K_n^+-member is $\leq k$, lest we have

$$\|K_n \geq_G K\| \geq \neg(y_n^0 \uparrow K_n),$$

violating $\|K_n \geq_G K\| = 0$. So let $y_n^1 \equiv y_n^0 \vee k$ for some such k; and, in general, iteratively define $y_n^{\alpha+1} \equiv y_n^\alpha \vee k$ for some k satisfying

$$k \leq \neg(y_n^\alpha \uparrow K_n),$$

and no K_n^+-member is $\leq k$, and define $y_n^\alpha \equiv \bigvee_{\beta<\alpha} y_n^\beta$ for limit ordinals α, until such time as $(y_n^\alpha \uparrow K_n) = 1$. Once this happens, set $y_n \equiv y_n^\alpha$. Note that $y_n \wedge (y_n^0 \uparrow K_n) = q_m$, so that there is a K_{n+1}^+-member that is $\leq q_m$.

It is then a matter of routine boolean algebra to verify that the induction hypothesis will hold at the next step, and that $Y \equiv \{y_n : n \geq 0\}$ will be as promised. □

Corollary 8.9 *A Cohen-forcing algebra K has a countable free subalgebra Y' that is dense in K.*

Invoke Lemma 8.8 for K with X empty. □

Corollary 8.10 *The Cohen-forcing algebra is unique up to isomorphism.*

Say A, B are Cohen-forcing algebras having (by Corollary 8.9) countably infinite free subalgebras A_0, B_0 that are dense in A and B respectively. Any isomorphism between sets of generators for A_0 and B_0 respectively will (as mentioned above) induce an isomorphism between A_0 and B_0; and because A_0 and B_0 are dense in A, B respectively, any isomorphism between A_0 and B_0 will extend uniquely to an isomorphism between A and B. □

Lemma 8.11 *The Cohen-forcing algebra satisfies ACSA-homogeneity (clause 1 of flexible homogeneity).*

This follows from Corollary 8.10 so long as each principal ideal $C \restriction c$ of each ACSA C of a Cohen-forcing algebra K is itself a Cohen-forcing algebra. And this is the case because if X is any countable dense subset of K, then $\{x \uparrow C : x \in X \cap (K \restriction c)\}$ is a countable dense subset of $C \restriction c$. □

Lemma 8.12 *There exists an ACSA C of the Cohen-forcing algebra K satisfying $1 \Vdash C \neq_G K$ (clause 2 of flexible homogeneity).*

Obtain from Corollary 8.9 a countable free subalgebra Y' of K that is dense in K; let $\{y_n : n \in \omega\}$ be a set of free generators for Y'. Let C_{even} denote the ACSA of K completely generated by $\{y_n : n \text{ even}\}$; and likewise for C_{odd} and $\{y_n : n \text{ odd}\}$. It is then straightforward to show that $1 \Vdash C_{even} \not\geq_G C_{odd}$, and therefore $1 \Vdash C_{even} \neq_G K$. □

Lemma 8.13 *If $1 \Vdash C <_G K$ then there exists an automorphism of K that is not an identity mapping but whose restriction to C is an identity mapping (clause (3) of flexible homogeneity).*

Use Lemma 8.8 to obtain a set of free generators of a free subalgebra that is dense in C. Call this set X; and use Lemma 8.8 again, with this X and K, to obtain a set $Y \subseteq K \setminus X$ such that $X \cup Y$ is a set of free generators for a free subalgebra F that is dense in K. The automorphism of $X \cup Y$ obtained by interchanging y and $\neg y$, for some chosen $y \in Y$, and leaving the other generators invariant, induces an automorphism of F, which in turn induces the desired automorphism of K. □

Lemma 8.14 *Each isomorphism ϕ between any $C, C' \in \mathsf{ACSAs}(K)$ satisfying*

$$1 \Vdash C, C' \neq_G K$$

extends to an automorphism of K (clause (4) of flexible homogeneity).

Fix C, C', ϕ. Obtain X, Y for C as in Lemma 8.13. By isomorphy, X's image under ϕ will be a set X' of free generators for a free subalgebra dense in C'. Use Lemma 8.8 with this X' and K to obtain Y'. Let θ be any bijection from the countable set Y of generators onto the countable set Y' of generators. It is straightforward to show that $\phi \cup \theta$ induces an automorphism of K. $\square$

Now Lemmas 8.11, 8.12, 8.13, 8.14 have established that all the clauses of the definition of flexible homogeneity hold for the Cohen-forcing algebra; so Theorem 8.7 is proved.

8.3 Random-real forcing does not work

A real number that is "random over L" in the sense established by R. Solovay is one that is interdefinable in a canonical way with a generic filter on a countably-completely-generated *measure algebra*. For the chief facts about measure algebras, see Chap. 30 of [7], or for more leisurely exposition, Vol. 3 of [4]; for more on random-real forcing, see the relevant parts of Chapters 15 and 26 in [7]. Our goal is to show that algebras of this kind are, like the Cohen algebra, homogeneously ACSA-complemented, and so succumb to Theorem 8.6.

A (strictly positive and normalized, or "probabilistic") *measure* on B is a real-valued function μ on B that satisfies

(i) $\mu(0) = 0$;

(ii) $\mu(b) > 0$ for all $b \in B^+$;

(iii) for all pairwise incompatible b_n, $n = 0, 1, \ldots$,

$$\mu(\bigvee_{n \geq 0} b_n) = \sum_{n \geq 0} \mu(b_n);$$

(iv) $\mu(1) = 1$.

A *measure algebra* is a complete boolean algebra B that carries a measure μ. (Actually we will equivocate on whether B or the pair (B, μ) "is" the measure algebra, much as we equivocate on whether B or the structure $\langle B, 0, 1, \wedge, \neg \rangle$ "is" the boolean algebra.)

Let R be an atomless, countably-completely-generated measure algebra, having measure μ. D. Maharam showed (see [7], Theorem 30.1, or [4], Chapter 33) that it is unique up to isomorphism, and that whenever it is isomorphic to some R' having measure μ', the witnessing isomorphism can be chosen to be measure-preserving, or in other words, to be an isometry with respect to μ and μ'.

Like all countably-completely-generated atomless boolean algebras, R has a countable free subalgebra that completely generates it; but unlike in the Cohen-forcing algebra case, no such free subalgebra is *dense* in R. To show flexible homogeneity for R we use a free subalgebra with a different property.

Definition. A set $\{b_i : i \in \omega\} \subset R$ is a *μ-independent generating subset* for R if its completion in R is (all of) R, and whenever x is the meet of n different b_i's and/or negations thereof (with all i's distinct), $\mu(x) = 2^{-n}$.

Lemma 8.15 *R has a μ-independent generating subset.*

This is probably best shown with the canonical example of measure algebras, namely the algebra of Borel subsets of the interval $(0,1)$ with their Lebesgue measures, modulo Lebesgue-null sets. It is well known that this algebra is completely generated by the rational subintervals of $(0,1)$; all these subintervals are completely generated just by those whose endpoints are terminating binary decimals, and these in turn are generated by the subsets of form

$$b_i \equiv \bigcup_{0 \leq n < 2^i} (\frac{2n}{2^{i+1}}, \frac{2n+1}{2^{i+1}}),$$

where $i \in \omega$. (Note that we have, as is customary in discussions of this algebra, stopped writing "modulo null sets" explicitly.) It is easily checked that these b_i give the required $\mu(x) = 2^{-n}$ property; it then suffices to note that Maharam's theorem guarantees an isometric isomorphism from this example algebra to whatever (R, μ) we fixed. $\square$

Definitions:

Fix a set of $b_i \in R$ as in Lemma 8.15.

Let $o_i \equiv b_{2i+1}$ (for $i \geq 0$) enumerate just R's odd generators, and let $e_i \equiv b_{2i}$ enumerate the even ones.

Let C_{even} denote the subalgebra of R completely generated by the e_i's, and similarly for C_{odd} with the o_i's.

We call subsets $X, Y \subseteq R$ *μ-independent* if $\mu(x \wedge y) = \mu(x)\mu(y)$ for all $x \in X, y \in Y$. Note that any pair C, D of μ-independent complete subalgebras will be each other's complements (as defined above) in $\overline{C \cup D}$.

Lemma 8.16 *If X, Y are μ-independent subalgebras of B then their completions $\overline{X}, \overline{Y}$ in B are μ-independent.*

We obtain $\overline{X}$ inductively by taking closures transfinitely many times, starting with $X_0 \equiv X$ and obtaining $X_1, X_2, \ldots, X_\alpha, \ldots$ as in Lemma 3.5, but with a slight difference here in the order of the operations under which we take closures. We must show that at each step, $X_{\alpha+1}$ remains μ-independent from Y. (This will hold trivially at limit α since we again define such X_α to be the union of previous stages.)

At stage $\alpha + 1$, then, consider X_α's closure under $\bigvee$:

$$X'_\alpha \equiv \{\bigvee Z : Z \subseteq X_\alpha\}. \tag{1}$$

For all $Z \subseteq X_\alpha$ and all $y \in Y$, we must show $\mu(y \wedge \bigvee Z) = \mu(y)\mu(\bigvee Z)$. We do this by showing that $\bigvee Z$ always equals the join of some countable set $\{z'_i\}$ of *mutually incompatible* elements of X_α. First note that some countable subset of Z will have the same join as Z: Enumerate Z as $\{z_\beta : \beta < \lambda\}$ (for some cardinal λ) and consider the "partial joins" $\bigvee_{\gamma<\beta} z_\gamma$ as β increases. If the partial joins increased at uncountably many β, their measure would increase uncountably many times; then $\mu(\bigvee Z)$ would be infinite, violating requirement (iv) of our definition of measure. So we may assume $\{z_i\}$ is an at-most-countable subset of Z with the same join.

Now iterate through the z_i starting from $i = 0$, defining corresponding z'_i by

$$z'_i \equiv z_i \wedge \neg \bigvee_{j<i} z_i.$$

Then the z'_i exist in X_α (which is closed under negation and finite joins), are mutually incompatible, and have the same join as the z_i and hence as Z.

To verify μ-independence at this step, observe that

$$\mu(y \wedge z'_i) = \mu(y)\mu(z'_i) \tag{*}$$

for all i, by the μ-independence of Y and X_α, and that

$$\begin{aligned}
\mu(y \wedge \bigvee Z) &= \mu(y \wedge \bigvee_i z_i') \\
&= \mu(\bigvee_i (y \wedge z_i')) && \text{[by distributivity]} \\
&= \sum_i \mu(y \wedge z_i') && \text{[by clause (iii) of defn. of measure]} \\
&= \mu(y) \sum_i \mu(z_i') && \text{[by (*)]} \\
&= \mu(y)\mu(\bigvee_i z_i') && \text{[by clause (iii) again]} \\
&= \mu(y)\mu(\bigvee Z).
\end{aligned}$$

Thus X'_α is μ-independent from Y.

Finally, let $X_{\alpha+1}$ be X'_α's closure under finite boolean operations. Fix some such operation and call its output b; we may write this as

$$b = \Psi(\bigvee Z_1, \ldots, \bigvee Z_n),$$

where Ψ is some boolean formula of n free variables, and each Z_m is a countable set of mutually incompatible X_α-members as above. Fix some enumeration of each Z_m, and and for all $j > 0$, let $Z_m^{\leq j}$ denote the join of the first j elements of Z_m's enumeration. For each m, the increasing sequence

$$\bigvee Z_m^{\leq 1} \leq \bigvee Z_m^{\leq 2} \leq \bigvee Z_m^{\leq 3} \leq \cdots$$

converges to $\bigvee Z_m$, and thus the terms' μ-values converge to $\mu(\bigvee Z_m)$. Furthermore, since there are only finitely many Z_m, this convergence is uniform for all of these sequences. More precisely, for all $\epsilon > 0$ there exists $k > 0$ such that for all $j > k$ and all $m \leq n$,

$$\mu((\bigvee Z_m) \wedge (\neg \bigvee Z_m^{\leq j})) < \epsilon.$$

Now define elements b_j by

$$b_j \equiv \Psi(\bigvee Z_1^{\leq j}, \ldots, \bigvee Z_n^{\leq j}).$$

By the uniform convergence just mentioned, $\mu(b_j)$ will converge to $\mu(b)$. (A rigorous proof of this would need something like an induction on the length of the formula

Ψ, but the details are not hard to fill in.) Since each b_j is a member of X_α, which is a subalgebra μ-independent from Y, we will have $\mu(b_j \wedge y) = \mu(b_j)\mu(y)$ for all $y \in Y$. It follows that in the limit we will have $\mu(b \wedge y) = \mu(b)\mu(y)$.

At some α this closure process will stop adding new elements and we will have $X_\alpha = \overline{X}$. This establishes the μ-independence of $\overline{X}$ and Y. That of $\overline{X}$ and $\overline{Y}$ follows when we carry out the same procedure above for Y, fixing $\overline{X}$ as the algebra from which we want to maintain independence. □

Lemma 8.17 *C_{even} and C_{odd} are μ-independent.*

Let O denote the closure of $\{o_i : i \in \omega\}$, and E that of $\{e_j : j \in \omega\}$, under *finite* boolean operations. We will establish that the subalgebras O and E are μ-independent; and the lemma will then follow from Lemma 8.16 since $C_{even} = \overline{E}$ and $C_{odd} = \overline{O}$.

Consider a boolean expression using k-many distinct o_i's (and no other elements), namely $o_{i_1}, o_{i_2}, \ldots, o_{i_k}$. It must evaluate to some element o of the finite subalgebra O_{fin} generated by just these o_i's. Any element of a finite boolean algebra is uniquely expressible as the join of finitely many atoms; in the case of our o, say m-many such atoms. And each atom a in O_{fin} has form

$$a = \pm o_{i_1} \wedge \pm o_{i_2} \wedge \cdots \wedge \pm o_{i_k},$$

where $\pm x$ means either x or $\neg x$.

Likewise, any E-member e will be a member of a similarly-defined E_{fin}, and be the join of finitely many (say n-many) E_{fin} atoms of form

$$b = \pm e_{j_1} \wedge \pm e_{j_2} \wedge \cdots \wedge \pm e_{j_l}.$$

We must verify $\mu(o \wedge e) = \mu(o)\mu(e)$. By the $\mu(x) = 2^{-n}$ propery of the o_i's and e_j's (Lemma 8.15), each atom a of O_{fin} has $\mu(a) = 2^{-k}$, each atom b of E_{fin} has $\mu(b) = 2^{-l}$; and $\mu(a \wedge b) = 2^{-k-l}$.

Now o is the join of m-many such (mutually incompatible) a's, and e is the join of n-many such (mutually incompatible) b's. The meet $o \wedge e$ is thus the "cross-product" of this set of a's with this set of b's; more precisely, $o \wedge e$ is the join of $m \times n$ many mutually incompatible elements of form $a \wedge b$. By additivity of μ on incompatible elements, then,

$$\mu(o \wedge e) = (m \times n)2^{-k-l} = (m \times 2^{-k})(n \times 2^{-l}) = \mu(o) \wedge \mu(e).\square$$

Lemma 8.18 *R is ACSA-homogeneous (clause (1) of flexible homogeneity).*

Consider any $C \in \mathsf{ACSAs}(R)$. It is straightforward to verify conditions (i)-(iv) above for μ's restriction $\mu \restriction C$. Except for (iv), the same is true for μ's restriction to any principal ideal $C \restriction c$; and by defining

$$\mu'(b) \equiv \mu(b)/\mu(c),$$

it is clear that $C \restriction c$, considered as a boolean algebra in its own right, carries the measure μ'. Thus by Maharam's uniqueness-up-to-isomorphism theorem, $C \restriction c$ is isomorphic to R. □

Lemma 8.19 *$1 \Vdash C_{even} <_G R$ (clause (2) of flexible homogeneity).*

To see that C_{even} is atomless, and hence a member of $\mathsf{ACSAs}(R)$, suppose it had an atom c. By definition, C^+_{even} could then have no members strictly $< c$, and *a fortiori* no such members of form $c \wedge b_i$ or $c \wedge \neg b_i$. Then for all even i, either $c \wedge b_i = 0$ or $c \wedge b_i = c$. It follows that either $c \leq b_i$ or $c \leq \neg b_i$. Thus for all n, c is $\leq$ the meet of n distinct elements of form b_i or $\neg b_i$; but then

$$\mu(c) \leq 2^{-n},$$

and as n can be arbitrarily large, $\mu(c)$ must be 0, contradicting requirement (ii) on measures.

We now show with a similar argument that $\|C_{even} <_G R\| = 1$. Suppose otherwise. In this case, since $\|C_{even} \leq_G R\| = 1$, we would have $\|C_{even} =_G R\| > 0$. Since $\|C_{odd} \leq_G R\| = 1$, we must have by the transitivity of $\leq_G$ that $\|C_{odd} \leq C_{even}\| > 0$. Thus by Lemma 4.9 there exists $b \in R^+$ such that $b \wedge C_{odd} \subseteq b \wedge C_{even}$.

This entails that for each generator o_i of C_{odd} there exists some $v_i \in C_{even}$ (not necessarily one of the e_i) such that $b \wedge o_i = b \wedge v_i$. By elementary boolean algebra, then,

$$b \leq (o_i \wedge v_i) \vee (\neg o_i \wedge \neg v_i).$$

Since this holds for all i, we have

$$b \leq \bigwedge_{i \in \omega} (o_i \wedge v_i) \vee (\neg o_i \wedge \neg v_i)$$

where each v_i is, again, some member of C_{even}.

We will show that the "finite partial meet" of the expression on the right side of this inequality, where i ranges from 0 only up to $n-1$, has measure 2^{-n}. It will then follow, just as in our proof of C_{even}'s atomlessness, that $\mu(b)$ must be 0, so that b must be 0, contradicting $b \in R^+$.

Define $\nu^+(i) \equiv o_i \wedge v_i$, and $\nu^-(i) \equiv \neg o_i \wedge \neg v_i$; $\nu^\pm(i)$ will denote one or the other. Note that $\nu^+(i) \wedge \nu^-(i) = 0$. Using this notation and basic distributivity, the partial meet

$$p_n \equiv \bigwedge_{i<n} (o_i \wedge v_i) \vee (\neg o_i \wedge \neg v_i)$$

is equal to the join of all 2^n elements of form

$$\nu^\pm(0) \wedge \nu^\pm(1) \wedge \ldots \wedge \nu^\pm(n-1),$$

where the $\pm$'s take all possible combinations of signs. These elements are clearly pairwise incompatible.

Now by regrouping we can write each element of the above form as

$$(\pm o_0 \wedge \pm o_1 \wedge \ldots \wedge \pm o_{n-1}) \wedge (\pm v_0 \wedge \ldots \wedge \pm v_{n-1}),$$

where the $\pm$'s for o_i and v_i have the same sign as the corresponding $\nu^\pm(i)$.

Now let $\{\mathbf{o}^m : 0 \leq m < 2^n\}$ consist of all 2^n elements of form

$$\pm o_0 \wedge \pm o_1 \wedge \ldots \wedge \pm o_{n-1},$$

and similarly for $\{\mathbf{v}^m : 0 \leq m < 2^n\}$, with the signs of the $\pm$'s aligned between every pair $\mathbf{o}^m$, $\mathbf{v}^m$ having the same index m.

The 2^n elements of form $\nu^\pm(0) \wedge \nu^\pm(1) \wedge \ldots \wedge \nu^\pm(n-1)$ are thus precisely the elements of form $\mathbf{o}^m \wedge \mathbf{v}^m$; so we have

$$p_n = \bigvee_{m<2^n} \mathbf{o}^m \wedge \mathbf{v}^m.$$

Now $\{\mathbf{o}^m : 0 \leq m < 2^n\}$ is a partition with $\mu(\mathbf{o}^m) = 2^{-n}$ for each m; $\{\mathbf{v}^m : 0 \leq m < 2^n\}$ is also a set of pairwise-incompatible elements whose μ-values sum to 1 (though their individual μ-values can vary). Since $\mu(\mathbf{o}^m \wedge \mathbf{v}^m) = \mu(\mathbf{o}^m)\mu(\mathbf{v}^m)$ for all m (Lemma 8.17), this gives (by basic combinatorial arithmetic)

$$\mu(p_n) = \mu(\bigvee_{m<2^n} \mathbf{o}^m \wedge \mathbf{v}^m) = 2^{-n}.$$

Since we noted above that $b \leq p_n$ for all n, and the μ-values of the p_n vanish, we must as promised have $b = 0$, contradicting $b \in R^+$. $\square$

To show that every C with $\|C <_G R\| = 1$ has a complement D in R, we use the following lemma from [7]:

Lemma 8.20 ([7], Lemma 30.5) *Let B be a measure algebra and let μ be a strictly positive measure on B [a class that includes measures as we defined them above]. Let C be a complete subalgebra of B and let ν be a measure on C such that $\nu(c) \leq \mu(c)$ for all $c \in C$. Assume that*

$$(*) \quad B \upharpoonright b \neq \{b \wedge c : c \in C\} \text{ for every } b \in B^+.$$

Then there exists some $b \in B$ such that

$$\nu(c) = \mu(b \wedge c) \text{ for all } c \in C. \square$$

Lemma 8.21 *If R is a measure algebra, and $\|C <_G R\| = 1$, and $0 < \epsilon < 1$, then there exists $b \in R$ such that for all $c \in C$, $\mu(c \wedge b) = \epsilon\mu(c)$.*

Apply Lemma 8.20 to our fixed measure algebra (R, μ) and some C such that $\|C <_G R\| = 1$, and define ν by $\nu(c) \equiv \epsilon\mu(c)$, so that clearly the $\nu(c) \leq \mu(c)$ requirement is satisfied. Note that requirement (*) on C is satisfied because it is equivalent to $\|C <_G R\| = 1$, by Lemma 4.9. Thus there exists $b \in R$ such that for all $c \in C$, $\epsilon\mu(c) = \nu(c) = \mu(b \wedge c)$. $\square$

Remark. Proofs of the next three results have been removed from this draft due to errors. If the stated results were shown to be false we would welcome it (it would make finding a model of self-collection much easier) but we fully expect that they are true and can be proved using Lemma 8.20 and the techniques of Section 333 of [4].

Lemma 8.22 *Every C with $\|C <_G R\| = 1$ has a complement D in R.*

Lemma 8.23 *R is homogeneously ACSA-complemented.*

Theorem 8.24 *No set of singly-generated continua in a random-real forcing extension of L self-constructs.*

9 Prospects For Self-Collection

Theorem 7.7 showed that (under Stipulation 3.10) a complete boolean algebra B will allow some $\mathcal{N}$ to self-collect into a singly-generated continuum just if B avoids having partial automorphisms of a particular kind. We now discuss ways that B might avoid having them.

A "rigid" approach.

The heavy-handed solution would be to find a B and $\mathcal{N}$ having an appropriate rigidity-like property. Consider the following construction of P. Hajek, as explained by R. Lubarsky in [10]:

> Hajek forces over L a sequence of reals $\langle a_n | n \in \omega \rangle$ which, when coded in some trivial manner into a real α, is a good candidate for the least upper bound of the c[onstructibility]-degrees of the a_n's. "Good candidate" means that any upper bound which does not collapse $\aleph_1$ also constructs the sequence. [...]
>
> Hajek's partial order is the ω-step iteration (with finite support) of the Jensen-Johnsbraten forcing [JJ]. The absoluteness comes about because any pair of JJ-generics (over the same ground model satisfying $V = L$) collapses $\aleph_1$. Also, a JJ-generic G over L is definable in $L[G]$. So if $\aleph_1^V = \aleph_1^L$ then the sequence of generics is definable as the ω-sequence $\langle a_0, a_1, \ldots \rangle$ such that a_{n+1} is the unique set satisfying the appropriate definition over $L[a_n]$.

From this construction we can define a set of distinct continua to serve as our $\mathcal{N}$:

$$\mathcal{N} = \{\mathbb{R}(a_n) : n \in \omega\}.$$

Now consider $\mathbb{R}(\alpha)$, where α is the real number cited above into which all the a_i have been "coded in some trivial manner" (like interleaving digits). We claim that all five requirements are met for $\mathcal{N}$ to self-collect into $\mathbb{R}(\alpha)$. Clearly $\mathbb{R}(\alpha) \notin \mathcal{N}$. $\mathcal{N}$ is linearly ordered (as $\mathbb{R}(a_0) \subset \mathbb{R}(a_1) \subset \ldots$) and so it is directed. By reversing the coding we can reconstruct the sequence of a_i's from α; thus $\mathcal{N} \in L(\mathbb{R}(\alpha))$ holds. For requirement (iv), note that any $x \in \mathbb{R}(\alpha)$ that constructs each a_n also constructs the sequence of all the a_n's (since x does not collapse $\aleph_1$), and so constructs α; thus $\mathbb{R}(x)$ cannot be a *proper* subset of $\mathbb{R}(\alpha)$. Finally, the sequence of a_i's is definable in $L(\bigcup \mathcal{N})$ (because the latter is $\subseteq L(\alpha)$ and so does not collapse $\aleph_1$); therefore we have $\alpha \in L(\bigcup \mathcal{N})$, so indeed $\mathbb{R}(\alpha) \in L(\bigcup \mathcal{N})$.

But this set $\mathcal{N}$ of continua is not self-*constructing*: for any finite $n > 0$, $\mathbb{R}(a_n)$'s predecessors do not self-collect into $\mathbb{R}(a_n)$, since

$$\mathbb{R}(\bigcup_{i<n} \mathbb{R}(a_i)) = \mathbb{R}(\mathbb{R}(a_{n-1})) = \mathbb{R}(a_{n-1}).$$

One avenue for seekng a self-constructing set of continua would thus be to modify this construction to obtain a linearly *but also densely* ordered set of continua, from a forcing algebra that, like Hajek's, is free of "bad" partial automorphisms. Our work in this direction has been inconclusive, but we are prepared to say that the task is difficult.

A "semi-rigid" approach.

A more subtle approach would be to seek a B that does have "bad" partial automorphisms in Φ_0, but does not "cleanly factor" into complementary subalgebras in the way that ruled out self-collection in Theorem 8.6, so that $\Phi_1 \neq \Phi_0$. One might try to construct a boolean algebra B that is "semi-rigid" in the following sense: B has nontrivial automorphisms, but for any atomless complete subalgebra $C \subseteq B$, an automorphism of B is uniquely determined by its action on C. To this end we have begun investigating B's defined from projection lattices of von Neumann algebras. This approach is also difficult, but seems more promising than the "rigid" approach described above. We hope to have a distributable version of [11] available soon, with some preliminary results.

10 Appendix: Development of Axioms From The Intuitive Description

Recall the definitions of *continuum* and of $\mathbb{R}(X)$ given at the outset. We wish to develop axioms for a set $\mathcal{F}$ of continua that answers to the following intuitive description:

A self-constructing set of continua grows gradually in such a way that each new continuum is just the closure under definable operations of the real numbers that had already arisen in the smaller continua, with neither "work from outside," nor any "background model of set theory," nor any "inexplicable novelty" involved in the course of its growth.

The requirement that each new continuum in $\mathcal{F}$ be "just the closure under definable operations of the real numbers that had already arisen in the smaller continua" has a natural formalization:

Axiom 0: $(\forall Y \in \mathcal{F})(Y = \mathbb{R}(\bigcup\{X \in \mathcal{F} : X \subset Y\}))$.

If in the context of $\mathcal{F}$ we define the *predecessors* of $Y \in \mathcal{F}$ to be those $\mathcal{F}$-members that are proper subsets of Y, we have the following noteworthy corollary to Axiom 0:

Corollary 10.1 *Any $Y \in \mathcal{F}, Y \neq \mathbb{R}(\emptyset)$, has infinitely many predecessors.*

For if $Y \neq \mathbb{R}(\emptyset)$ were a counterexample, there would exist a counterexample $X \neq \mathbb{R}(\emptyset)$ (either a predecessor of Y or Y itself) that either has no predecessors or has $\mathbb{R}(\emptyset)$ as its *only* predecessor. In the first case Axiom 0 implies $X = \mathbb{R}(\emptyset)$, a contradiction; in the second, we have $X = \mathbb{R}(\bigcup\{\mathbb{R}(\emptyset)\}) = \mathbb{R}(\mathbb{R}(\emptyset)) = \mathbb{R}(\emptyset)$, the same contradiction. □

We next try to understand what it might mean formally for $\mathcal{F}$ to grow "gradually" with no "work from outside" involved in the course of its growth. The clearest way to do this is to take a relatively simple model of Axiom 0 and pinpoint the ways it intuitively *fails* to grow this way.

Definitions. A *singly-generated* continuum X is one that satisfies $X = \mathbb{R}(x)$ for some real number x. As in Example 2.1, let G be a generic filter for Cohen forcing (see [7], Chapter 15); let $\mathcal{C}_S$ be the set of singly-generated continua in $L[G]$; let $\mathcal{C}_{\neg S}$ be the set whose members are $\mathbb{R}(\emptyset)$ and all non-singly-generated continua in $L[G]$.

(Note that by Lemma 11.5 we will always have $L(G) = L[G]$ in the cases under consideration in this section; as noted in the discussion of that Fact, the "parentheses" version is the relevant one for us in general, but the "brackets" version is the natural one in the context of forcing extensions.)

To see that $\mathcal{C}_S$ satisfies Axiom 0, and to weigh how much it deserves to be called "self-constructing," we need some facts about its structure.

Fact 10.2 *Each inner ZFC model of a Cohen forcing extension $L[G]$ has form $L(x)$ for some real number $x \in L[G]$, so (in light of Lemma 11.5) $\mathcal{C}_S$ can equivalently be defined as the set of continua of $L[G]$'s inner ZFC models.*

Thus $L[G]$'s inner ZFC models are all generated by individual real numbers, which can be partitioned into equivalence classes according to which model they generate. These classes are known as "degrees of constructibility" and it is mainly in this guise that this structure has been studied (see [1] for a thorough overview). $\mathcal{C}_S$'s most salient features are these:

Fact 10.3 *$\mathcal{C}_S$ is* densely ordered *([1], Corollary 1.2) in the sense that it has infinitely many intermediate members between any strictly ordered pair $X \subset Y$ of its members, and*

Fact 10.4 *$\mathcal{C}_S$ is* complemented *([1], Theorem 1.1) in the sense that for every pair $X, Y \in \mathcal{C}_S$ such that $X \subseteq Y$, there exists $X' \in \mathcal{C}_S$ such that $\mathbb{R}(X \cup X') = Y$ and $\mathbb{R}(X \cap X') = \mathbb{R}(\emptyset)$; furthermore, when x_1, x_2 are any real numbers witnessing $X = \mathbb{R}(x_1)$ and $X' = \mathbb{R}(x_2)$, we have $Y = \mathbb{R}(\{x_1, x_2\})$.*

Facts 10.3 and 10.4 entail that $\mathcal{C}_S$ does indeed satisfy Axiom 0: every $Y \in \mathcal{C}_S$ is constructible from the union of all its predecessors, because it is constructible (aside from the case $Y = \mathbb{R}(\emptyset)$) just from a *pair* x_1, x_2 of reals chosen from among its predecessors. But we argue that this construction from pairs of predecessors violates the spirit of "gradual growth" in our intuitive description of self-construction. In particular, when Y, X, X' are as in Fact 10.4 and none of them equals $\mathbb{R}(\emptyset)$, we make the following accusations:

—Y is *predetermined* by its predecessors X and X', since although $Y = \mathbb{R}(X \cup X')$ there exist by Fact 10.3 "intermediate" $Z, Z' \in \mathcal{C}_S$ satisfying $X \subset Z \subset Y$ and $X' \subset Z' \subset Y$; the growth of new real numbers from X to Z and from X' to Z' thus seems superfluous for the construction of Y.

—Y has a set of *mutually unrelated* predecessors that construct it: X and X' are unrelated to each other in the strong sense that they are mutually generic, and appealing to this pair's existence in a self-constructing family as justification for Y's existence there evokes "outside work" that is hand-picking X and X', and yoking them together to take their constructive closure.

—Y is *overdetermined* by its predecessors, being constructed independently as $Y = \mathbb{R}(\{x_1, x_2\})$ and as $Y = \mathbb{R}(\{x_3, x_4\})$ for some some real x_1, x_2, x_3, x_4 belonging respectively to four of Y's predecessor continua X_1, X_2, X_3, X_4, such that no X_i includes any of the others. A proof here requires more facts than just 10.3 and 10.4 but follows easily from this example: let $y \subseteq \omega$ be a Cohen real that witnesses Y's being singly generated, so $Y = \mathbb{R}(y)$; let x_1 be the set of y's even members, and x_2 be $y \setminus x_1$; let x_3 be the set of y's members that are divisible by three, and let x_4 be $y \setminus x_3$.

To satisfy our intuitive description, a member X of $\mathcal{F}$ ought instead to be constructed through the coming-together of smaller continua in $\mathcal{F}$ that naturally form a collection and require vanishingly little work to close.

Since we know (Corollary 10.1) that each non-least $X \in \mathcal{F}$ must have infinitely many predecessors, we suggest that continua "naturally form a collection" if they have already come together in $\mathcal{F}$ in all their finite combinations. Furthermore if the continuum X which such a collection generates is to be an $\mathcal{F}$-member, it ought to be possible to reconstruct the collection from X; otherwise some ambient model of set theory outside $L(X)$ would have been needed to specify the collection whose union constructed X. Finally, X ought to be the "limit" or "least upper bound" of the collection in the sense that no strictly smaller continuum $Y \subset X$ contains each member of the collection; otherwise the difference between Y and X could be regarded as a "gap" that would require externally-supplied "work" to surmount. If a set $\mathcal{N}$ of continua met these conditions, the emergence of $X = \mathbb{R}(\bigcup \mathcal{N})$ from $\mathcal{N}$ could be seen as the final step of a process each of whose infinitely many previous steps had already been completed, and thus as requiring vanishingly little work.

We propose the following formalization of this suggestion.

Definitions:

A set $\mathcal{N}$ is *directed* (with respect to the inclusion ordering) if $X, Y \in \mathcal{N} \Rightarrow (\exists Z \in \mathcal{N})(X, Y \subseteq Z)$. (This formalizes the idea that $\mathcal{N}$'s members have "already come together in all finite combinations.")

A set $\mathcal{N}$ of continua *self-collects into* X if all of the following hold:

(i) $X \notin \mathcal{N}$;

(ii) $\mathcal{N}$ is directed;

(iii) $\mathcal{N} \in L(X)$;

(iv) $(\neg\exists x \in X)(\bigcup\mathcal{N} \subseteq \mathbb{R}(x) \subset X)$;
(v) $X = \mathbb{R}(\bigcup\mathcal{N})$.

Draft Self-Collection Axiom: $(\forall X \in \mathcal{F})$(some $\mathcal{N} \subseteq \mathcal{F}$ self-collects into X).

Remark 1. Clause (iv) demands that no singly-generated continuum lie "between" $\mathcal{N}$ and X; arguably we should demand that no *non*-singly-generated continuum lie there either, but until this issue can be shown to cause problems, we are content to ignore it for the sake of simplicity.

Remark 2. Clause (iii) of self-collection is needed to ensure that the Self-Collection Axiom will be expressible as a first-order statement, without tacitly invoking an "ambient ZF model," which would violate our intuitive description. One reason we cannot ignore such matters is discussed under "The Absoluteness Issue" below.

Remark 3. We could combine Axiom 0 with our Draft Self-Collection Axiom by requiring that the set of *all* X's predecessors in $\mathcal{F}$ self-collect into X. This would simplify matters. The reason we do not do this is that the set of predecessors of each $\mathcal{F}$-member would then have to be directed, and our intuitive description seems not to demand this: if a proper subset $\mathcal{N}$ of X's predecessors in $\mathcal{F}$ self-collects into X and constructs other continua $Y, Z \subseteq X$ as "by-products" (so to speak), such that Y and Z have no common superset in $\mathcal{N}$, this does not obviously violate our intuition about self-collection, even if Y and Z happen themselves to be members of $\mathcal{F}$.

Returning to our example $\mathcal{C}_S$, might the continuum $\mathbb{R}(x) \in \mathcal{C}_S$ of an unconstructible real x also equal $\mathbb{R}(\bigcup\mathcal{N})$ for some subset $\mathcal{N}$ of its predecessors that self-collects? The answer, given by our Theorem 8.6, is no:

Fact 10.5 *Given any $\mathbb{R}(x) \in \mathcal{C}_S \setminus \mathbb{R}(\emptyset)$, no subset of $\mathbb{R}(x)$'s predecessors self-collects into $\mathbb{R}(x)$; therefore $\mathcal{C}_S$ does not satisfy Draft Self-Collection Axiom.*

Now $\mathbb{R}(x)$ *does* have increasing sequences (of length ω_1) of predecessors in $\mathcal{C}_S$ that meet requirements (i) through (iv) of self-collection, and are unbounded in the set of all its predecessors. So we may well ask, if $\mathcal{N}$ is such a sequence, what its $\mathbb{R}(\bigcup\mathcal{N})$ could be, if not $\mathbb{R}(x)$. The answer is that it is the continuum of an inner model that violates AC (the Axiom of Choice) and is thus not a member of $\mathcal{C}_S$.

This raises the question: in proposing $\mathcal{C}_S$ as a candidate to be a self-constructing set, did we have any justification for excluding this non-singly-generated continuum $\mathbb{R}(\bigcup\mathcal{N})$ from it in the first place? More generally: *which* subsets of a self-constructing set $\mathcal{F}$ get "considered as collections" so as to beget new continua in $\mathcal{F}$

via union-and-constructive-closure? If $\mathcal{F}$ had finite cardinality, we could have appealed to a maximality principle: *every* $\mathcal{F}$-subset undergoes the union-and-closure operation, resulting in an $\mathcal{F}$-member. But since $\mathcal{F}$ has infinite cardinality (Corollary 10.1) there is no clear fact of the matter about what subsets of $\mathcal{F}$ are available to undergo this. In general, it would depend on what "outer" (or "background" or "ambient") ZF model we took $\mathcal{F}$ to reside in—and any appeal to such a model would directly violate the intuition we began with.

Since naive maximality principles are excluded, arbitrariness may seem inescapable. For if $\mathcal{F}$ has subsets that "could" be considered as collections this way, but aren't—so that the continua they construct "would" belong to $\mathcal{F}$, but don't—there is evidently someone on the outside choosing which collections to consider as such, and $\mathcal{F}$ cannot be regarded as *self*-constructing.

A partial solution we propose for this problem is to take the conditions of our Draft Self-Collection Axiom as necessary *and sufficient* for an $\mathcal{F}$-subset to generate an $\mathcal{F}$-member. In other words, not only must each $\mathcal{F}$-member be constructed by a self-collecting subset of $\mathcal{F}$, but *every such subset* constructs an $\mathcal{F}$-member.

Self-Collection Axiom: $(\forall X \in L(\mathcal{F}))(X \in \mathcal{F} \iff \text{some } \mathcal{N} \subseteq \mathcal{F} \text{ self-collects into } X)$.

Remark 1. The empty set self-collects trivially into $\mathbb{R}(\emptyset)$, which is thus required by the Self-Collection Axiom to be a member of any self-constructing family. One could keep $\mathbb{R}(\emptyset)$ out by adding $\mathcal{N} \neq \emptyset$ as another clause in the definition of self-collection, but it is not clear than a net gain in convenience would result.

Remark 2. This version of the Self-Collection Axiom may not *fully* resolve the issue of non-absoluteness; see "The absoluteness issue" below.

Finally, our intuitive notion requires a principle of foundation to keep real numbers from emerging through "inexplicable novelty" or "work from outside" during the course of $\mathcal{F}$'s growth. To see how this problem can occur, we will consider a different subset $\mathcal{C}_{\neg S}$ (also defined above, in Example 2.1) of the continua in a Cohen forcing extension $L[G]$.

Fact 10.6 *$\mathcal{C}_{\neg S}$ can be defined equivalently as*

$$\{X \in L[G] : (\exists \mathcal{N} \subseteq \mathcal{C}_S)(\mathcal{N} \textit{ self-collects into } X)\}.$$

We have already remarked that the empty set self-collects into the constructible continuum $\mathbb{R}(\emptyset)$. The $L(X)$ of any other continuum $X \in L[G]$ satisfies AC iff it has form $\mathbb{R}(x)$ for some real x (Fact 10.2): if it does, Fact 10.5 states that no such

$\mathcal{N}$ self-collects into it; otherwise, it is straightforward to verify that $\mathcal{N}$ defined as $\{\mathbb{R}(x) : x \in X\}$ satisfies the definition of self-collection into X. □

$\mathcal{C}_{\neg S}$ satisfies Axiom 0 and the Self-Collection Axiom—but in an underhanded way, as we will now see, with the help of the following lemma:

Lemma 10.7 *For any distinct $Y, Y' \in \mathcal{C}_S$ such that $Y \subset Y'$, there exists in $L(Y')$ an unbounded subset $\mathcal{M}$ of Y''s predecessors (in $\mathcal{C}_S$) that has Y as a member and self-collects into $\bigcup \mathcal{M}$; thus by Fact 10.6, $\bigcup \mathcal{M} \in \mathcal{C}_{\neg S}$.*

By Fact 10.3, $L(Y')$ has no immediate predecessors in $\mathcal{C}_S$. Since Y' is singly generated, the Axiom of Choice holds in $L(Y')$. Thus Zorn's lemma entails there the existence of a maximal directed subset $\mathcal{M}$ of Y''s predecessors in $\mathcal{C}_S$, with $Y \in \mathcal{M}$. $\mathcal{M}$ clearly meets all but the last requirement for self-collection into Y'. But Fact 10.5 implies that $Y' \not\subseteq \mathbb{R}(\bigcup \mathcal{M})$; therefore

$$\mathcal{N} \equiv \{\mathbb{R}(x) : x \in \mathbb{R}(\bigcup \mathcal{M})\}$$

is a subset of Y''s predecessors such that $\mathcal{M} \subseteq \mathcal{N}$ and $\bigcup \mathcal{N} = \mathbb{R}(\bigcup \mathcal{M})$. It is clear that $\mathcal{N}$ is directed (since $x, y \in \mathbb{R}(\bigcup \mathcal{M})$ implies $z \in \mathbb{R}(\bigcup \mathcal{M})$ for some z interdefinable with $\{x, y\}$), so we must have $\mathcal{M} = \mathcal{N}$, lest $\mathcal{N}$ contradict $\mathcal{M}$'s maximality. Thus $\mathbb{R}(\bigcup \mathcal{M}) = \bigcup \mathcal{M}$, and it is easy to verify all the clauses required for $\mathcal{M}$ to self-collect into $\bigcup \mathcal{M}$. □

Lemma 10.8 *Within the Cohen forcing extension $L[G]$, given an arbitrary set X, some subset of $\mathcal{C}_S$ self-collects into X if and only if some subset of $\mathcal{C}_{\neg S}$ does.*

First suppose $\mathcal{N} \subseteq \mathcal{C}_{\neg S}$ self-collects into X. Define

$$\mathcal{N}' \equiv \{\mathbb{R}(x) : x \in \bigcup \mathcal{N}\} \subseteq \mathcal{C}_S;$$

we will verify that $\mathcal{N}'$ meets all the clauses of the definition of self-collection into X.

(i): $X \notin \mathcal{N}'$ because otherwise $X \in \mathcal{N}$: if $X = \mathbb{R}(x)$ for some $x \in \bigcup \mathcal{N}$, then this x is a member of some $Y \in \mathcal{N}$, so $\mathbb{R}(x) \subseteq Y$; but $Y \subseteq X = \mathbb{R}(x)$, so $Y = X$.

(ii): to verify directedness, take any $\mathbb{R}(x), \mathbb{R}(y) \in \mathcal{N}'$; we know $x, y \in \bigcup \mathcal{N}$, so by $\mathcal{N}$'s directedness, there must exist $Z \in \mathcal{N}$ such that $x, y \in Z$. Then letting $z \in Z$ be any real number interdefinable with $\{x, y\}$, we have $\mathbb{R}(z) \in \mathcal{N}'$, and $\mathbb{R}(x), \mathbb{R}(y) \subseteq \mathbb{R}(z)$.

(iii): follows from the transitivity of the relation "x constructs y." $\mathcal{N}'$ is constructible from $\mathcal{N}$ so the supposition that $\mathcal{N}$ self-collects into X, and in particular that it satisfies the $\mathcal{N} \in L(X)$ clause (iii) of self-collection into X, implies that $\mathcal{N}'$ satisfies this clause too.

(iv): plainly a continuum will be a counterexample to this clause for $\mathcal{N}$ if and only if it is a counterexample for $\mathcal{N}'$.

(v): It is clear that $\bigcup \mathcal{N}' = \bigcup \mathcal{N}$, so certainly $\mathbb{R}(\bigcup \mathcal{N}') = X$.

Conversely, suppose that $\mathcal{N}$ is now a subset of $\mathcal{C}_S$ that self-collects into X. No Y can be maximal in $\mathcal{N}$ since the latter is directed and has no bound in X's singly-generated sub-continua, so for each $Y \in \mathcal{N}$ choose a larger continuum $Y' \in \mathcal{N}$ such that $Y \subset Y'$. By Lemma 10.7 there exists a $\mathcal{C}_{\neg S}$-member M such that $Y \subseteq M \subseteq Y'$. Thus if we define

$$\mathcal{N}' \equiv \{M \in \mathcal{C}_{\neg S} : (\exists Y' \in \mathcal{N})(M \subseteq Y')\},$$

we will have $\bigcup \mathcal{N}' = \bigcup \mathcal{N}$. The other clauses can then be shown as in the proof of this lemma's other direction. □

The above lemma along with Facts 10.5 and 10.6 entails:

Fact 10.9 *$\mathcal{C}_{\neg S}$ satisfies Axiom 0 and the Self-Collection Axiom.* □

Nonetheless, we cannot regard $\mathcal{C}_{\neg S}$ as self-constructing, for it cannot account for *how* any of its real numbers "got constructed." Choose any non-least $X \in \mathcal{C}_{\neg S}$ and any unconstructible real $x \in X$. What accounts for x's presence in X? Could the process of closing X's predecessors' union under definable operations have newly generated x? No, x was already a member of one these predecessors. Consider: X is a non-least $\mathcal{C}_{\neg S}$-member and thus cannot be equal to $\mathbb{R}(x)$; so there exists $y \in X \setminus \mathbb{R}(x)$. $\mathbb{R}(\{x, y\})$ is the least continuum having x and y as members, so $\mathbb{R}(\{x, y\}) \subset X$. Now apply Lemma 10.7 with $\mathbb{R}(x)$ as Y and $\mathbb{R}(\{x, y\})$ as Y'; this yields a $\mathcal{C}_{\neg S}$-member that has x but is a proper sub-continuum of X. Thus any $X \in \mathcal{C}_{\neg S}$ that has x as a member can point to one of its predecessors and say "well, x was already possible there," and so on ad infinitum. The explanatory buck stops nowhere, and x seems like a case of inexplicable novelty, belying the idea that $\mathcal{C}_{\neg S}$ is self-constructing.

To rule such situations out, it would certainly suffice to use the following axiom: "Every real appearing in some member of $\mathcal{F}$ appears at a least such member." This may be unnecessarily strong, though, ruling out some inoffensive cases. Suppose $\mathcal{F}$-members X and Y are least for containing reals x and y respectively, and that neither $x \in Y$ nor $y \in X$; would anything mysterious necessarily be involved if there were two distinct $\mathcal{F}$-members Z_1 and Z_2 that were minimal for containing some z interdefinable with $\{x, y\}$? Our intuition is that there would not; a continuum that is minimal in $\mathcal{F}$ for having z as a member represents a way of building z out of "less complex" reals; the fact that two such ways exist is not at odds with the idea that $\mathcal{F}$ is self-constructing. The idea is violated not when there are *multiple* ways of constructing a real out of less complex reals, but when there are *no* ways. Hence the following axiom:

Foundation Axiom: $(\forall X \in \mathcal{F})(\forall x \in X)(\exists Y \in \mathcal{F})(x \in Y \subseteq X$ and $(\forall Z \in \mathcal{F})(Z \subset Y \Rightarrow x \notin Z))$.

Fact 10.10 *$\mathcal{C}_S$ satisfies our Foundation Axiom but $\mathcal{C}_{\neg S}$ does not.*

$\mathcal{C}_S$ satisfies our Foundation Axiom because for every $x \in \bigcup \mathcal{C}_S$, $\mathbb{R}(x)$ is the least $\mathcal{C}_S$-member having x as a member. $\mathcal{C}_{\neg S}$'s failure to satisfy the Axiom is clear from our discussion leading up to our statement of it. □

Dropping Axiom 0; Defining Self-Construction

It can perhaps be shown that Axiom 0 is implied by our Self-Collection axiom. We have not been able to do so. However, even if there were a counterexample — a continuum $X \in \mathcal{F}$ some subset of whose $\mathcal{F}$-predecessors self-collect into it, but which is not constructed by the union of *all* its $\mathcal{F}$-predecessors — we could choose to regard it as a pathological case of self-collection, but not one that should dissuade us from calling $\mathcal{F}$ self-constructing (assuming it satisfies the Self-Collection and Foundation axioms). Thus, in the interest of simplicity, we prefer to omit Axiom 0 from our definition of self-construction.

Definition. A set $\mathcal{F}$ of continua is *self-constructing* if it satisfies the Self-Collection and Foundation Axioms.

The Absoluteness Issue

Although we will take the Self-Collection and Foundation axioms developed above to define a self-constructing set of continua, there is an issue that threatens to weaken their claim to do so.

To see the issue, suppose $\mathcal{F} \subset \mathcal{F}'$ are distinct sets of continua such that $\mathcal{F}$ satisfies the axioms in $L(\mathcal{F})$, and $\mathcal{F}'$ satisfies the axioms in $L(\mathcal{F}')$. There might exist in $L(\mathcal{F}')$ (or at least it is not obvious why there couldn't exist) a continuum Y, and a subset $\mathcal{N} \subset \mathcal{F}$, neither of which exist in $L(\mathcal{F})$, such that $\mathcal{N}$ self-collects into Y. Y would therefore be a member of $\mathcal{F}'$. But why was it not already a member of $\mathcal{F}$? We in $L(\mathcal{F}')$ "now realize" that the members of $\mathcal{N}$ were ready and waiting to self-collect into Y, but for some reason they did not. This recalls what we said earlier: If $\mathcal{F}$ has subsets that "could" be considered as collections this way, but aren't, so that the continua they construct "would" belong to $\mathcal{F}$, but don't, there is evidently someone on the outside choosing which collections to consider as such, and $\mathcal{F}$ cannot be regarded as *self*-constructing.

What we would like to have, then, is something about $\mathcal{F}$ guaranteeing that no new self-collecting subsets of it can ever be "discovered" later. Note that this requirement is not (at least in its most straightforward form) an axiom, because it is not just claiming that something holds in a particular ZF model. For present purposes, we will simply define a *strongly sealed* self-constructing set as one whose structure underwrites a guarantee of this type, and table the matter of *how* its structure might do this.

Definition. A self-constructing family $\mathcal{F}$ of continua is *strongly sealed* if its satisfaction of the Self-Collection Axiom is upwards-absolute in the sense that it will still hold when "$(\forall X \in L(\mathcal{F}))$" is replaced with "$(\forall X \in L(\mathcal{F}'))$" for any self-constructing family $\mathcal{F}' \supseteq \mathcal{F}$.

We choose to call this property "strongly sealed" in order to leave the term "sealed" available for a property that is weaker and perhaps better aligned with our intuitive notion. To see why the strongly-sealed property may be too strong, consider that cases of self-collection $X = \mathbb{R}(\mathcal{N})$ can fall into two categories. When $X = \bigcup \mathcal{N}$, no new real numbers are generated by self-collection, and we call this *sterile* self-collection. (Note all the self-collection in $\mathcal{C}_{\neg S}$ was sterile.) When X is strictly larger than $\bigcup \mathcal{N}$, we call it *generative* self-collection since new real numbers are generated. We submit that the "discovery" in $\mathcal{F}'$ of new sub-collections of $\mathcal{F}$ that self-collect in the *sterile* way would be innocuous.

Definition. A self-constructing family $\mathcal{F}$ of continua is *sealed* if for no larger self-constructing family $\mathcal{F}' \supseteq \mathcal{F}$ (in any ZF model containing $L(\mathcal{F})$) can there exist $\mathcal{N} \in L(\mathcal{F}')$, $\mathcal{N} \subseteq \mathcal{F}$, such that $\mathcal{N}$ *generatively* self-collects into some $X \notin \mathcal{F}$.

Since this definition of the property of being *sealed* quantifies over sets "in any ZF model containing $L(\mathcal{F})$," there is no straightforward way to axiomatize it. However, we restrict our attention in the body of this paper to self-constructing families that can by obtained by forcing over the constructible universe L; within the scope of this restriction, it may be possible to axiomatize the "sealed" property by quantifying over all the complete boolean algebras in L that could yield families larger than $\mathcal{F}$ if they were to be used for forcing. We will not pursue this here, though.

11 Appendix: Review of Relative Constructibility, Basic Facts

The best exposition of constructibility theory is probably the first pair of chapters in K. Devlin's book [3]. The theory is developed in a similar way but more tersely in Chapter 13 of T. Jech's standard set-theory text [7]. We will not reproduce this material here, for, as Devlin writes in his preface, "Constructibility theory is plagued with a large number of extremely detailed and potentially tedious arguments, involving such matters as investigating the exact logical complexity of various notions of set theory." What we will do here is recall enough of the theory to make it plain that a few basic facts hold that will be fundamental to our work.

11.1 The Constructible Universe L

The axioms of ZF, Zermelo-Frankel set theory with no Choice axiom, capture faithfully but incompletely the intuitive notion of the hierarchy of all well-founded sets (assuming there is one unique such hierarchy). In particular ZF gives an incomplete answer to the question of what subsets an infinite set can have, and its answer is still incomplete if a Choice axiom (or its negation) is added. Of course ZF's Power Set axiom ensures that "all" subsets of X themselves form a set, and its Separation axioms ensure that any subset of X that can be defined (by a sentence of the formal language of set theory using finitely many set parameters) will be among "all" these subsets. But ZF leaves open the question of what other subsets, not definable in this way, might also exist.

The parsimonious position that no other subsets should exist is embodied in the *constructible universe* L. L is built by transfinite induction using the function $\mathsf{Def}(X)$, which yields the set of all definable subsets of a given set X. More precisely, $\mathsf{Def}(X)$ is the set of all sets of form

$$\{x \in X : \Psi^X(x, c_1, \ldots, c_n)\}$$

where Ψ is a proposition of the language of set theory having one free variable x, and $c_1, \ldots, c_n$ are some constants in X, and Ψ^X means Ψ's relativization to X (any unbounded quantifiers in Ψ are constrained to range over X). L is then defined as the union of all stages L_α of the *constructive hierarchy*:

$L_0 \equiv \emptyset$;

$L_{\alpha+1} \equiv \mathsf{Def}(L_\alpha)$;

$L_\alpha \equiv \bigcup_{\beta<\alpha} L_\beta$ when α is a limit ordinal;

$L \equiv \bigcup_{\alpha \in Ord} L_\alpha$.

L is a model of ZF ([3], Theorem 1.2; [7], Theorem 13.3) and also of the Axiom of Choice ([3], Theorem 3.8; [7], Theorem 13.18).

The construction of L relativizes quite simply to $L(X)$ for arbitrary sets X: specifically, we "seed" the hierarchy at stage 0 by letting $L_0(X)$ be the transitive closure of $\{X\}$. ($\mathsf{TrCl}(\{X\})$ is taken as the seed, rather than X, so that the resulting model will be transitive, which is a requirement to count as an "inner" model of ZF as that term is generally defined.) We then carry out the construction the same way:

$L_0(X) \equiv \mathsf{TrCl}(\{X\})$;

$L_{\alpha+1}(X) \equiv \mathsf{Def}(L_\alpha(X))$;

$L_\alpha(X) \equiv \bigcup_{\beta<\alpha} L_\beta(X)$ for limit α;

$L(X) \equiv \bigcup_{\alpha \in Ord} L_\alpha(X)$.

This definition can be found in [7] as Definition 13.24, where it is noted that $L(X)$ is the smallest inner model of ZF having X as a member. To *prove* that L is the smallest inner model ([7], Theorem 13.16) and that $L(X)$ is the smallest inner model having X as a member, we must show that they can be defined in an *absolute* way.

11.2 Absoluteness of L and $L(X)$

Definition. When $\Psi(x, Y)$ is a proposition of the language of set theory having exactly one free variable (x) and exactly one constant symbol (Y), we call Ψ a *Y-basic proposition*; when Y is understood to denote some specific set, we call $\Psi(x, Y)$ a *Y-basic predicate*; and if furthermore the expression

$$\{x : \Psi(x, Y)\}$$

defines the same set in any standard inner ZF model (having Y as a member!) in which it is evaluated, we call $\Psi(x, Y)$ a *Y-absolute basic predicate.*

Fact 11.1 *There is a Y-basic proposition Δ such that when Y is chosen to refer to any set, $\Delta(x, Y)$ is a Y-absolute basic predicate and $\{x : \Delta(x, Y)\}$ equals $\mathsf{Def}(Y)$ as defined above.*

To establish this fact we will simply reference [7], Lemma 13.14 and the initial segment of Chapter 13 leading up to it; or see [3], Lemma 2.4, and the pages leading up to it (where it is the formula $D(v, u)$ that formalizes Def). The terminology used in both references is slightly different from ours, but in both cases the formula's absoluteness is proved by showing that it has a low enough level of logical complexity, in terms of its quantifiers. We will not enter into the details here. □

The next step in showing the absoluteness of L and $L(Y)$ is to formalize the transfinite construction used in their definitions, and to show that each step preserves absoluteness. We will abstract the proof somewhat so that it can be reused for other hierarchies in our paper.

Lemma 11.2 (Absolute Definability Of Transfinite Hierarchies) *Suppose $\Omega(x', Z)$ is a Z-absolute basic predicate for some fixed set Z, and that $\Psi(x'', (Y, Z, \beta))$ is a (Y, Z, β)-absolute basic predicate for any ordinal β and any set Y. Then for all ordinals α, the following is a (Z, α)-absolute basic predicate:*

$(\exists f)(f$ *is a function;*

$\mathrm{dom}(f) = \alpha + 1$;

$f(0) = \{x' : \Omega(x', Z)\}$;

$(\forall \beta < \alpha)(f(\beta + 1) = \{x'' : \Psi(x'', (f(\beta), Z, \beta))\})$;

$(\forall \beta \leq \alpha)(\beta$ *is a limit ordinal* $\Rightarrow f(\beta) = \bigcup_{\gamma<\beta} f(\gamma))$;

and $x \in f(\alpha)$ *).*

This also holds when $\bigcup_{\gamma<\beta}$ is replaced with $\bigcap_{\gamma<\beta}$.

Note first that the function f will in fact always exist, by ZF's Axiom Schema of Replacement. Now suppose that for some α, the stated predicate fails to be (Z, α)-absolute. Then some *least* α witnesses this failure. But α cannot be 0 (since $\Omega(x', Z)$ is $(Z, 0)$-absolute), cannot be a successor ordinal (since then $f(\alpha-1)$ would be $(Z, \alpha - 1)$-absolute by the induction hypothesis and $\Psi(x'', (f(\alpha - 1), Z, \alpha - 1)$ is $(f(\alpha - 1), Z, \alpha - 1)$-absolute, and cannot be a limit ordinal (since the union and intersection operations are absolute). □

Lemma 11.3 *For any ordinal α and any set Z, L_α is definable by an α-absolute basic predicate, and $L_\alpha(Z)$ is definable by a (Z, α)-absolute basic predicate.*

In both cases we plug propositions $\Omega(\cdots)$ and $\Psi(\cdots)$ into the template of Lemma 11.2. In the L_α case, let Z be $\emptyset$, and for $\Omega(x', Z)$ use "$x' \in \emptyset$" or any other always-false assertion, so that L_0 will be the empty set; and for $\Psi(x'', (Y, Z, \beta))$ use the proposition $\Delta(x'', Y)$ that (by Fact 11.1) defines $\mathsf{Def}(Y)$ and is a Y-absolute basic predicate for any Y.

Given any set Z, we obtain the Z-absolute predicate for $L_\alpha(Z)$ the same way, but using the given Z instead of $\emptyset$, and for $\Omega(x', Z)$ using the assertion "$x' \in \mathsf{TrCl}(\{Z\})$," which is clearly definable by a Z-absolute predicate. $\square$

From this lemma, the absoluteness of L and of $L(Z)$ follow: for any sets x and Z, the assertion $(\exists\alpha)(x \in L_\alpha(Z))$ will be either true in all standard inner ZF models, or false in all such models.

The absoluteness of L allows us to use L-members freely in any Y-absolute basic predicate without affecting its Y-absoluteness. More precisely, if $(c_1, \ldots c_n)$ are an ordered n-tuple of sets in L, and for some Y and Ψ, $\Psi(x, (Y, c_1, \ldots, c_n))$ is a $(Y, c_1, \ldots, c_n)$-absolute basic predicate, then

$$\{x : \Psi(x, (Y, c_1, \ldots, c_n))\}$$

defines the same set in *any standard inner model having Y as a member* — we need not specify "... having $(Y, c_1, \ldots, c_n)$ as a member" since every such model has $c_1, \ldots, c_n$ as members. Likewise, if any c_i is a member of $\mathsf{TrCl}(\{Y\})$, we need not specify "... having c_i as a member" once we have specified "having Y as a member." Therefore we may define an easier-to-use but equivalent absoluteness condition (which is the one given in Section 3.2):

Definition. When y is any set, a *y-predicate* is a proposition $\Psi(x, c_1, \ldots, c_n)$ of the language of set theory, having one free variable x and finitely many constant symbols $c_1, \ldots, c_n$, considered together with fixed referents for the c_i *chosen out of* $L \cup \mathsf{TrCl}(\{y\})$. A y-predicate $\Psi(x, c_1, \ldots, c_n)$ is *y-absolute* if

$$\{x : \Psi(x, c_1, \ldots, c_n)\}$$

evaluates to the same set in any standard inner ZF model having y as a member.

Lemma 11.4 *Lemma 11.2 still holds when $\Omega(x', Z)$ is allowed to be a Z-absolute (no longer necessarily basic) predicate $\Omega(x', c_1, \ldots, c_n)$, and $\Psi(x'', (Y, Z, \beta))$ is allowed to be a (Y, Z, β)-absolute predicate $\Psi(x'', c'_1, \ldots, c'_m)$; the predicate guaranteed by the lemma to be (Z, α)-absolute for all α will now be a (Z, α)-absolute predicate using the c_i's and c'_j's as constants.* $\square$

11.3 Proofs of constructibility lemmas from Section 3.2

Proof of Lemma 3.1: This follows from Lemma 11.3, which states the same thing in terms of absolute *basic* predicates. $\square$

Proof of Lemma 3.4:

The lemma's claim is: $z \in L(y)$ if and only if $z = \{x : \Psi(x, c_1, \ldots, c_n)\}$ for some y-absolute predicate $\Psi(x, c_1, \ldots, c_n)$. Note that we consider each c_i's referent as part of the predicate, and we will freely conflate the c_i's with their referents.

Clearly the "if" direction holds since $L(y)$ is a normal ZF model having y as a member.

For "only if," suppse $z \in L(y)$. By definition of the $L(y)$ hierarchy, z must appear as member of some lowest level $L_\alpha(y)$, and α cannot be a limit ordinal because limit levels of the hierarchy are just unions of previous levels. If $\alpha = 0$, then by the definition of $L_0(y)$, we have $z \in \mathsf{TrCl}(\{y\})$; thus z itself can legally be used as (the referent of) a constant in a y-absolute predicate. So we can simply let $\Psi(x, z)$ be the proposition "$x \in z$," and we have our y-absolute predicate that defines z.

The remaining case is that α is a successor ordinal. In this case, $L_\alpha(y) = \mathsf{Def}(L_{\alpha-1}(y))$, and therefore z must be defined by some formula Ψ relativized to $L_{\alpha-1}(y)$, using only a finite number of $L_{\alpha-1}(y)$-members $c_1, \ldots, c_n$ as constants.

Now we know (Lemma 3.1) that $L_{\alpha-1}(y)$ is itself definable by a y-absolute predicate using constants only for $\alpha - 1$ and $\mathsf{TrCl}(\{y\})$. Say $L_{\alpha-1}(y)$ is so defined by $\Delta(x, \alpha - 1, \mathsf{TrCl}(\{y\}))$. Then the predicate $\Omega(x, \alpha - 1, \mathsf{TrCl}(\{y\}), c_1, \ldots, c_n)$ defined by

$$\Omega(x) \iff (\exists M)((\forall v)(v \in M \iff \Delta(v, \alpha - 1, \mathsf{TrCl}(\{y\}))), \text{ and } \Psi^M(x, c_1, \ldots, c_n))$$

is a predicate (of free variable x) that defines z and is absolute for standard inner ZF models having $\mathsf{TrCl}(\{y\})$ and all the c_i as members. $\Omega(x)$ need not, however, be a y-predicate (as we have defined the term), since the c_i need not be members of $L \cup \mathsf{TrCl}(\{y\})$.

The key point, though, is that even if (the referent of) some c_i is not a member of $L \cup \mathsf{TrCl}(\{y\})$, it will still be a member of $L(y)$, and will have appeared first at some lower level $L_{\beta+1}(y)$. The task now is to replace any uses of such a constant c_i in Ω with a formula that was used to define its referent at step $\beta + 1$ as a subset of $L_\beta(y)$. We will not formalize this replacement; we will simply note that it can be done, and that the result is a predicate $\Omega'(x)$ that will have a new constant c'_1 denoting β, and finitely many $c'_1, \ldots, c'_m$ denoting members of L_β, as new constant parameters. But the predicate will retain its absoluteness, and will still define z, and will no longer use c_i.

We now wish to iterate this replacement to remove any other c_i's — and now also any c'_i's — that are not members of of $L \cup \mathsf{TrCl}(\{y\})$. Doing so yields a tree of constants whose referents first appear at stages $L_\beta(y)$, and whose "replacement constants" appear at *lower levels* of the hierarchy. Because constants are replaced by formulas using constants from *lower* ordinal stages, the replacement process must terminate, with a final finite set of constants all in $L \cup \mathsf{TrCl}(\{y\})$, lest there be an infinite descending chain of ordinals. The final replacement predicate that uses all these constants is y-absolute and defines z. □

11.4 The "square-brackets" version of relative constructibility

There is a second way to relativize the L construction to a set, namely the $L[X]$ construction (note the square brackets). This is in fact more commonly seen than the "parentheses" version, possibly because it is a more natural generalization of the generic extension construction $L[G]$ where G is a generic filter. (When G is a filter, $L[G]$ will indeed be the same whether calculated by the rules of generic extensions or as the hierarchy we will define just below, but we will not prove this equivalence here.)

$L[X]$ is defined as the union of all stages $L_\alpha[X]$, which are themselves defined exactly as the L_α's, but using an "augmented" definable-subsets operation, $\mathsf{Def}_X(Y)$. This operation is in turn defined just like $\mathsf{Def}(Y)$, except that its propositions Ψ may use, in addition to the usual syntax of set theory, an additional one-place predicate "$\underline{v \in X}$" that is true just if v is a member of X. In other words, we do not throw X directly into this construction as a set; instead we allow the definable operations used in the construction to consult a sort of oracle, to ask whether a given set v would be a member of X, if X existed. (And indeed, in certain cases, X will not be a member of $L[X]$.) So defined, $L[X]$ is a standard inner model of ZF, and moreover of ZFC.

Happily, in most of the cases we will consider, we will have $L[X] = L(X)$:

Lemma 11.5 *$L[X] = L(X)$ for any set $X \subseteq L$.*

Fix X such that $X \subseteq L$. The key point is that this ensures $X \in L[X]$, as follows. There must exist α such that $X \subseteq L_\alpha$, lest X be a proper class. By L_α's absoluteness, it is a member of the standard inner ZF model $L[X]$, and therefore of some $L_\beta[X]$. Since $L_\beta[X]$ is transitive, $X \subseteq L_\beta[X]$. It is then a member of $\mathsf{Def}_X(L_\beta[X])$, since it is definable within $L_\beta[X]$ using the "oracle" predicate, namely as $\{v \in L_\beta[X] : \underline{v \in X}\}$. Thus $X \in L_{\beta+1}[X] \subseteq L[X]$.

Now, as remarked above, $L(X)$ is the smallest ZF model (that is well-founded, transitive, and has all the ordinals) that has X as a member; since we have established $L[X]$ is such a model, we must have $L(X) \subseteq L[X]$.

In the other direction we cite Theorem 13.22 (iii) of [7]: If M is an inner model of ZF such that $X \cap M \in M$, then $L[X] \subseteq M$. $L(X)$ is such an M; therefore $L[X] \subseteq L(X)$. (Alternatively we could proceed from the observation that when X is a member of our model, we can replicate the "oracular" predicate $\underline{v \in X}$ with the "normal" membership predicate $v \in X$, and thereby reconstruct the Def_X hierarchy in our model.) □

Full proof of Lemma 5.1:

The "only if" direction of the lemma is what needs to be proved. Assume $G \in L(\Theta)$ and (invoking Lemma 3.4) that G is definable by a Θ-absolute predicate as

$$G = \{x : \Delta(x, c_1, \ldots, c_n)\}.$$

We will now show that there exists a Θ-absolute predicate of the more narrowly-defined form demanded in the lemma's statement, that defines the same set (namely G).

We claim it suffices to show that each c_i used by the given Δ can itself be defined by a Θ-absolute predicate of the more narrowly-defined form. Suppose for all $i \in \{1, \ldots, n\}$ that c_i were defined by a Θ-absolute predicate as

$$c_i = \{x : \Delta_i(x, \Theta, G \cap C_i, c'_i)\}$$

for some C_i with $G \cap C_i \in \bigcup \Theta$, and some constant $c'_i \in L$. By clause (ii) of Lemma 4.2, the ACSA C' defined by

$$C' \equiv \overline{C_1 \cup \ldots \cup C_n}$$

satisfies $G \cap C' \in \bigcup \Theta$. Thus for all i, since $C_i \in L$, we can alter $\Delta_i(\ldots)$ so that it continues to be a Θ-absolute predicate defining c_i, but instead of taking $G \cap C_i$ as a constant parameter, takes $G \cap C'$ and C_i as constant parameters, and uses "$(G \cap C') \cap C_i$" wherever the proposition Δ_i used "$G \cap C_i$". This yields a Θ-absolute predicate

$$\Delta'_i(x, \Theta, G \cap C', C_i, c'_i)$$

that defines c_i and has the more narrowly-defined form. Once this is done for all the c_i, we can similarly transform the given $\Delta(\ldots)$ that defines G, replacing each c_i with the Δ'_i expression that defines it, yielding

$$G = \{x : \Delta'(x, \Theta, G \cap C', C_1, \ldots, C_n, c'_1, \ldots, c'_n)\},$$

where $\Delta'(\ldots)$ has the required narrower form.

To show that each c_i can be defined by some Δ'_i this way, we consider four cases. By definition of Θ-absolute predicate, c_i must be a member of L and/or $\mathsf{TrCl}(\{\Theta\})$.

Case 1 : $c_i \in L$. In this case set $c'_i = c_i$, let C_i be arbitrary, and let $\Delta'_i(x, \ldots)$ simply be the proposition $\{x \in c'_i\}$.

Case 2: $c_i = \Theta$. In this set $c'_i = \emptyset$, let C_i be arbitrary, and let $\Delta'_i(x, \ldots)$ simply be the proposition $\{x \in \Theta\}$.

Case 3: $c_i = \theta(X) \in \Theta$ for some continuum X. By Lemma 4.5, c_i will have a member $G \cap C_i$ for some C_i, and c_i will be the set of all copies of this $G \cap C_i$ on subalgebras in $\mathsf{ACSAs}(B)$, via functions in $\mathsf{ParAut}(B)$. Since $\mathsf{ACSAs}(B)$ and $\mathsf{ParAut}(B)$ are constructible sets, c_i is moreover *absolutely* definable from $G \cap C_i$ this way, as

$$c_i = \{x : \Delta'_i(x, \Theta, G \cap C', C_i, \{\mathsf{ACSAs}(B), \mathsf{ParAut}(B)\})\},$$

for an appropriate proposition Δ'_i that formally defines being a copy. Thus we can choose this C_i and Δ'_i at our step i, and set $c'_i = \{\mathsf{ACSAs}(B), \mathsf{ParAut}(B)\}$.

Case 4: $c_i = F \in \theta(X) \in \Theta$ for some continuum X and some filter F. This is handled in essentially the same way as Case 3; here c'_i can be $\{C, \phi\}$, where C is the ACSA on which F is an ultrafilter and ϕ is a particular $\mathsf{ParAut}(B)$-member that copies $G \cap C_i$ to F.

Note that all the members of filters F that can appear in Case 4 are B-members and hence constructible; thus all other $c_i \in \mathsf{TrCl}(\{\Theta\})$ would have fallen under Case 1, so there are no more cases. □

Proof of $C^{+b} \in \mathsf{ACSAs}(B)$ in Lemma 6.1:

Note first that C^{+b} is atomless because if $(c_1 \wedge b) \vee (c_2 \wedge \neg b)$ were an atom then either c_1 or c_2 would be an atom of C, which is not the case. If C^{+b} is a complete subalgebra of B then it is countably-completely-generated: since some countable subset Q of C completely generates C, clearly $Q \cup \{b\}$ completely generates C^{+b}.

To verify that C^{+b} is indeed a complete subalgebra of B, we show that it is closed under negation and arbitrary joins. (Closure under meets will follow from $\bigwedge X = \neg(\bigvee\{\neg e : e \in X\})$.)

C^{+b} is closed under negation because

$\neg((c_1 \wedge b) \vee (c_2 \wedge \neg b)) =$

$\neg(c_1 \wedge b) \wedge \neg(c_2 \wedge \neg b) =$

$(\neg c_1 \vee \neg b) \wedge (\neg c_2 \vee b) =$

$(\neg c_1 \wedge \neg c_2) \vee (\neg c_1 \wedge b) \vee (\neg b \wedge \neg c_2) \vee (\neg b \wedge b) =$

$(\neg c_1 \wedge \neg c_2) \vee (\neg c_1 \wedge b) \vee (\neg b \wedge \neg c_2)$.

Since $(\neg c_1 \wedge \neg c_2)$ implies $(\neg c_1 \wedge b) \vee (\neg b \wedge \neg c_2)$, this string of equivlent expressions is also equivalent to $(\neg c_1 \wedge b) \vee (\neg b \wedge \neg c_2)$, which is $\in C^{+b}$ since it has the correct form.

C^{+b} is closed under $\bigvee$ because

$\bigvee_{\alpha<\beta}((c_{1,\alpha} \wedge b) \vee (c_{2,\alpha} \wedge \neg b)) =$

$(\bigvee_{\alpha<\beta}(c_{1,\alpha} \wedge b)) \vee (\bigvee_{\alpha<\beta}(c_{2,\alpha} \wedge \neg b)) =$

$(b \wedge \bigvee_{\alpha<\beta} c_{1,\alpha}) \vee (\neg b \wedge \bigvee_{\alpha<\beta} c_{2,\alpha}) \in C^{+b}$.

(We are appealing to the basic distributivity property of complete boolean algebras here.) $\square$

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